

# AN EFFECTIVE FIELD THEORY MODEL FOR DIFFERENTIAL ELLIPTIC COHOMOLOGY AT THE TATE CURVE

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**ABSTRACT.** We construct a model for differential elliptic cohomology at the Tate curve whose cocycles are families of 2-dimensional effective supersymmetric field theories. A geometrically-motivated modularity condition requires partition functions to take values in  $\mathrm{TMF}(X) \otimes \mathbb{C}$ . Cocycles satisfying this condition yield classes in a 24-periodic differential cohomology theory whose coefficients are the ring of integral modular forms. We construct examples of cocycles from families of string manifolds and principal bundles with structure group the monster.

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## 1. INTRODUCTION

Since Witten and Segal’s groundbreaking papers [Wit87, Wit88, Seg88] there has been a tantalizing yet elusive connection between elliptic cohomology and 2-dimensional field theories. Just as the 1-dimensional physics of Dirac operators is related to K-theory, the dream has been that a 2-dimensional generalization would lead to a similarly rich collection of elliptic cohomology classes. This paper makes progress by constructing a modular version of differential elliptic cohomology at the Tate curve whose cocycles are 2|1-dimensional *effective supersymmetric field theories*. These are a variation on the *supersymmetric* field theories studied by Stolz and Teichner [ST04, ST11] and Cheung [Che08] that incorporate ideas from *effective* field theories we learned from Kitaev [Kit09] and Costello [Cos11].

Effective field theories are designed to simplify the functional analysis of quantum field theory by way of an energy cutoff. This typically reduces problems in analysis to ones in (finite-dimensional) linear algebra. In the second paragraph of the introduction to [Wit88], Witten remarked that a “cutoff version of the nonlinear sigma model would be adequate” to capture all the physics relevant for his topological conjectures. Our definition of effective supersymmetric field theory seeks to build a framework for such cutoff versions of the relevant sigma models.

The primary focus of this paper is to compare 2|1-dimensional effective field theories to elliptic cohomology by way of the topological  $q$ -expansion principle. The starting point is elliptic cohomology at the Tate curve.

**Theorem 1.1.** *There is a natural isomorphism of rings,*

$$\widehat{K}_{\mathrm{Tate}}(X) \cong 2|1\text{-AFT}_{\mathrm{eff}}(X)/\sim$$

between differential elliptic cohomology at the Tate curve of  $X$  and stable isomorphism classes of 2|1-dimensional effective annular field theories over  $X$ .

Part of the data of a differential refinement is a differential form-valued curvature map for  $2|1\text{-AFT}_{\text{eff}}(X)$  that comes from evaluating an effective annular field theory on a moduli stack of tori over  $X$  with specified meridian. This is called the *partition function*. We can take the subcategory of effective annular field theories whose partition function pulls back from the underlying stack of tori *without* a chosen meridian, or more generally, pulls back from a section of the  $n^{\text{th}}$  tensor power of a certain line bundle over this stack. We denote this subcategory by  $2|1\text{-EFT}_{\text{eff}}^n(X)$  and call its objects *degree  $n$ , 2|1-dimensional effective Euclidean field theories over  $X$* . These are very close in spirit to Stolz and Teichner's degree  $n$  field theories; indeed, they determine field theories defined on a subcategory of the Stolz–Teichner bordism category. In tandem with Theorem 1.1, objects in  $2|1\text{-EFT}_{\text{eff}}^{2n}(X)$  define cocycles for a cohomology theory we call KMF that is a modular version of elliptic cohomology at the Tate curve. We describe KMF in homotopy-theoretic terms in subsection 1.2. For now we note that KMF is a 24-periodic cohomology theory whose coefficients are the ring of integral modular forms, and it receives a map from the cohomology theory of topological modular forms (TMF).

**Theorem 1.2.** *There is a natural isomorphism of abelian groups*

$$\widehat{\text{KMF}}^{2n}(X) \cong 2|1\text{-EFT}_{\text{eff}}^{2n}(X)$$

*between stable isomorphism classes of degree  $2n$ , 2|1-dimensional effective Euclidean field theories over  $X$  and classes in differential  $\text{KMF}^{2n}$  of  $X$ .*

We provide examples of cocycles in differential KMF constructed from families of string manifolds and principal bundles for the monster group in Section 5. Although Theorem 1.2 stops short of giving a geometric model of TMF, it does elucidate aspects of the emerging connections between 2|1-dimensional supersymmetric field theories, differential geometry, and elliptic cohomology.

The difference between KMF and TMF is a plethora of 2- and 3-torsion. On the physical side of Theorem 1.2, our effective field theories lack many of the standard bells and whistles of a 2-dimensional field theory, e.g., they do not have a state-field correspondence or an operator product expansion. It will be very interesting to understand these geometric refinements of KMF classes homotopically, and we have organized our definitions in ways that will be amenable to some of these questions. It would be truly amazing if certain refinements detected obstructions to lifting classes along the map  $\text{TMF}(X) \rightarrow \text{KMF}(X)$ .

To provide rationale and motivation for the constructions leading to Theorem 1.1, we formulate and prove a similar statement for K-theory.

**Theorem 1.3.** *There is a natural isomorphism of rings,*

$$\widehat{\text{K}}(X) \cong 1|1\text{-EFT}_{\text{eff}}(X),$$

*between stable isomorphism classes of 1|1-dimensional effective field theories over  $X$  and the differential K-theory of  $X$ .*

Related constructions of odd K-theory, odd Tate K-theory and odd KMF are a bit more nuanced, and so we focus on the even case in this paper. The problem comes from a well-known issue in constructing *finite-dimensional* models for odd K-theory via Clifford linear operators. We briefly describe odd variants of our cocycles that map to  $\text{K}_{\text{Tate}}^{\text{odd}}(X)$  in Remark 4.3.

**1.1. Elliptic cohomology and topological  $q$ -expansion.** One starting point for understanding the ring of topological modular forms is through its map to integral modular forms. This map of rings lifts to one of spectra, as we describe by way of the topological

$q$ -expansion principle [Lau99],

$$(1) \quad \begin{array}{ccc} \mathrm{TMF}(X) & \xrightarrow{\mathrm{ev}_{\mathrm{Tate}}} & \mathrm{K}_{\mathrm{Tate}}(X) \\ \otimes \mathbb{C} \downarrow & & \downarrow \mathrm{ch} \\ \mathrm{TMF}(X) \otimes \mathbb{C} & \xrightarrow{q\text{-expand}} & \mathrm{H}(X; \mathbb{C})[[q]][q^{-1}]. \end{array}$$

The top horizontal map is Miller's elliptic character [Mil89] that evaluates TMF in a formal punctured neighborhood of the Tate curve, and the left vertical map is the generalized Chern character of TMF from tensoring over  $\mathbb{Z}$  with  $\mathbb{C}$ . To describe the remaining maps we identify elliptic cohomology at the Tate curve with ordinary K-theory with coefficients in powers of  $q$ ,  $\mathrm{K}_{\mathrm{Tate}}(X) \cong \mathrm{K}(X)[[q]][q^{-1}]$ , and TMF with complex coefficients with ordinary cohomology with coefficients in the graded  $\mathbb{C}$ -algebra of weak modular forms,  $\mathrm{TMF}(X) \otimes \mathbb{C} \cong \mathrm{H}(X; \mathrm{MF})$ . Then the right vertical arrow is induced by the Chern character in K-theory, and the lower horizontal arrow is determined by the  $q$ -expansion of weak modular forms.

Let KMF denote the homotopy pullback (in a category of ring spectra) of the part of the diagram (1) that excludes TMF; the coefficient ring of KMF is the ring of integral modular forms. Hence the canonical map  $\mathrm{TMF} \rightarrow \mathrm{KMF}$  of spectra lifts the ring map from topological forms to integral modular forms. Roughly, one might think of KMF as the height 1 truncation of TMF. It seems plausible that KMF is the restriction of the sheaf of spectra  $\mathcal{O}^{\mathrm{top}}$  to elliptic curves whose formal group law is not super singular at any prime, though we do not know a characterization of KMF in these terms.

In Corollary 4.11, we compare effective field theories to a related variant of (1)

$$(2) \quad \begin{array}{ccc} \mathrm{Tmf}(X) & \xrightarrow{\mathrm{ev}_{\mathrm{Tate}}} & \mathrm{K}(X)[[q]] \\ \otimes \mathbb{C} \downarrow & & \downarrow \mathrm{ch} \\ \mathrm{Tmf}(X) \otimes \mathbb{C} & \xrightarrow{q\text{-expand}} & \mathrm{H}(X; \mathbb{C})[[q]] \end{array}$$

where Tmf is the the universal elliptic cohomology theory over the Deligne–Mumford compactification of the moduli stack of elliptic curves. This is the version of topological modular forms related to the ring of ordinary modular forms wherein the discriminant  $\Delta$  is *not* invertible; similarly, Tmf is not periodic. The relevant version of Tate K-theory in (2) evaluates on a formal (non-punctured) neighborhood of the Tate curve. We denote the analogous homotopy pullback for (2) as Kmf.

**1.2.  $q$ -expansion for 2|1-Euclidean field theories.** For a smooth manifold  $X$ , Stolz and Teichner have defined a bordism category denoted  $2|1\text{-EBord}(X)$  whose objects are closed 1|1-dimensional supermanifolds with a map to  $X$ , and whose morphisms are compact 2|1-dimensional super Euclidean manifolds with a map to  $X$ . Disjoint union of bordisms gives a symmetric monoidal structure. Their main conjecture is the existence of a 2-categorical refinement (i.e., a fully-extended bordism category) such that there is a natural ring isomorphism

$$\mathrm{TMF}(X) \cong \mathrm{Fun}^{\otimes}(2|1\text{-EBord}(X), \mathcal{V}) / \sim \quad (\text{conjectural}),$$

for a symmetric monoidal 2-category  $\mathcal{V}$  that deloops (topological) vector spaces.<sup>1</sup> Inspired by (1), we extract simpler pieces from  $2|1\text{-EBord}(X)$  and investigate an analogous square,

$$(3) \quad \begin{array}{ccc} \text{Fun}^\otimes(2|1\text{-EBord}(X), \mathcal{V}) & \xrightarrow{\text{Res}_{\text{annuli}}} & \text{Fun}(\text{Ann}_0^{2|1}(X), \text{Vect}) \\ \text{Res}_{\text{tori}} \downarrow & & \downarrow Z \\ \text{Fun}(\mathcal{M}^{2|1}(X), \mathbb{C}) & \xrightarrow{\text{forget}} & \text{Fun}(\widetilde{\mathcal{M}}^{2|1}(X), \mathbb{C}) \end{array}$$

where  $\text{Res}_{\text{annuli}}$  and  $\text{Res}_{\text{tori}}$  are restrictions to subcategories of  $2|1\text{-EBord}(X)$  consisting of annuli and tori respectively,  $Z$  is the partition function (which in this case is a trace), and the remaining arrow is induced by a forgetful functor. In more detail, the subcategory  $\text{Ann}_0^{2|1}(X) \subset 2|1\text{-EBord}(X)$  consists of super annuli (as bordisms between super circles) whose map to  $X$  factors through a super point;  $\mathcal{M}^{2|1}(X) \subset 2|1\text{-EBord}(X)$  consists of super tori (as bordisms from the empty set to itself) whose maps to  $X$  factor through a super point, and  $\widetilde{\mathcal{M}}^{2|1}(X) \subset \text{Ann}_0^{2|1}(X)$  consists of super tori with a chosen meridian and a map to  $X$  that factors through the super point. The lower horizontal arrow is induced by the forgetful functor  $\widetilde{\mathcal{M}}^{2|1}(X) \rightarrow \mathcal{M}^{2|1}(X)$  that forgets the meridian. These bordisms have energy zero in the  $2|1$ -dimensional sigma model for any choice of metric on  $X$ , and so we call them *energy zero annuli* and *energy zero tori*. A key benefit of the energy zero condition is that the associated categories have finite-dimensional presentations.

The arrows in (3) can be viewed as extracting lower categorical data:  $\text{Ann}_0^{2|1}(X)$  is a 1-category and  $\mathcal{M}^{2|1}(X)$  is a 0-category, and hence the appropriate target categories are  $\text{Vect}$  and  $\mathbb{C}$ , respectively, in (3). Functors from a 0-category to  $\mathbb{C}$  are just *functions*, and in [BE13] we showed the set of these functions (with some conditions) give cocycles for  $\text{TMF}(X) \otimes \mathbb{C}$ . In this paper we focus on understanding the category of functors  $\text{Ann}_0^{2|1}(X) \rightarrow \text{Vect}$ ; Theorem 1.1 characterizes isomorphism classes with additional effective field theoretic data, and Theorem 1.2 studies compatibility with functions on  $\mathcal{M}^{2|1}(X)$ .

The idea of focusing attention on annuli goes back to Segal's famous notes on conformal field theory [Seg04]. In his thesis [Che08], Pokman Cheung uses ideas from the Stolz–Teichner program to incorporate supersymmetry to Segal's annuli and constructs a *space* of annular field theories (over  $X = \text{pt}$ ) representing Tate K-theory. The terminology of *annular field theory* is borrowed from Cheung. Although our version shares similar ideas, its output is a *category of cocycles on  $X$* . For subsequent differential-geometric constructions, the picture presented by cocycles illuminates features that aren't so clear from Cheung's classifying space.

Following Segal, Stolz–Teichner and Cheung, the obvious definitions for a category of representations of super annuli over  $X$  run into nuanced analysis almost immediately. Ideas from effective field theories allow us to control these analytical issues.

**1.3. Effective supersymmetric mechanics and K-theory.** To give a flavor of the effective field theory techniques we employ, we explain how effective supersymmetric quantum mechanics (i.e.,  $1|1$ -dimensional effective field theory) can be incorporated into a description of K-theory, loosely following some ideas of Kitaev [Kit09]. For now we will be vague about our precise notion of  $1|1$ -dimensional field theory, but we have in mind the barebones ingredients of quantum mechanics: a vector space of states and a time-evolution operator.

An even-dimensional compact spin manifold  $M$  determines a quantum mechanical system whose  $\mathbb{Z}/2$ -graded space of states is sections of the spinor bundle, and time evolution operator is  $\exp(-t\mathcal{D}^2)$  for  $\mathcal{D}$  the (self-adjoint, odd) Dirac operator. The  $\lambda$ -eigenspace of  $\mathcal{D}^2$  consists of the *energy  $\lambda$  states*. A *low-energy effective theory* takes the states of energy less than a *cutoff*  $\lambda > 0$  with time-evolution the restriction of  $\exp(-t\mathcal{D}^2)$ . Since  $M$  is compact,

<sup>1</sup>There is a refinement relating  $\text{TMF}^n(X)$  and *degree  $n$  twisted field theories* that requires a double delooping of  $\text{Vect}$  to a 3-category.

this subspace of states is finite-dimensional. The difference between spaces of states for cutoffs  $\lambda$  and  $\lambda'$  is the addition of a finite-dimensional vector space on which  $\mathcal{D}$  is *invertible*. Hence, varying the cutoff stabilizes a low-energy effective field theory by a vector space of super dimension zero. Therefore, the original quantum theory determines an underlying K-theory class that can be represented by *any* cutoff theory, and varying the cutoff ranges through these representatives.

From a bundle of spin manifolds with base  $X$  we obtain a *family* of quantum systems: the fiberwise spinor bundle and Dirac operator determine a space of states and time-evolution operator as above. If there is a  $\lambda > 0$  not in the spectrum of the square of any Dirac operator in the family, then  $\lambda$  gives a cutoff and an associated family of effective field theories. Different cutoffs will differ by a vector bundle on which the Dirac operators are invertible, and hence there is a single underlying K-theory class for any family of effective theories extracted from the initial data. More generally, by a Lemma of Mischenko–Fomenko [MF79] (see also Lemma 7.11 of [FL10]) *any* family of Dirac operators has a finite-dimensional subbundle of the fiberwise spinors containing the kernel of the family of Dirac operators, and this subbundle defines a family of effective field theories over  $X$ .

There is one deficiency in the passage to finite-dimensions: the differential form-valued Chern character of the original family of Dirac operators (e.g., using the Bismut super connection) need not equal the Chern character determined by the cutoff theory. This Chern character is the *partition function* of the family of field theories (compare [Han08]), and is an all-important observable in supersymmetric field theory. We can encode changes in the partition function using another important idea from effective field theory: integration over higher energy states. Concretely, the Bismut–Cheeger  $\eta$ -form [BC89] of a Bismut super connection  $\mathbb{A}$

$$\eta := \int_0^\infty \text{Tr} \left( \frac{d\mathbb{A}(\lambda)}{d\lambda} e^{-\mathbb{A}(\lambda)^2} \right) ds$$

mediates between the Chern character of the original infinite-dimensional super connection and the restricted finite-dimensional one, where  $\mathbb{A}(\lambda)$  is an interpolation between these super connections. We shall explain in Section 2.3 how this family can be viewed as arising from the renormalization group flow on the category of 1|1-dimensional field theories, where  $\lambda$  indexes an energy scale. In this way, we view the integral defining the Bismut–Cheeger form as a version of Wilson’s integrating out higher energy states.

The ethos of effective field theory is that one never need work directly with the infinite-dimensional objects. Instead we analyze finite-dimensional cutoff versions and relations between different cutoffs that both modify the space of states (by stabilization) and the partition function (through variants of Chern–Simons forms). These ideas translate readily to the 2|1-dimensional setting and lead to our model for differential Tate K-theory.

**1.4. Ingredients for a low-energy effective field theory.** Our low-energy effective field theories in dimension 1|1 and 2|1 come from two pieces of data: (1) a functor  $F$  out of a flavor of bordism category over  $X$  and (2) auxiliary data that modifies the partition function. The functor  $F$  turns out to be equivalent to familiar differential geometric data (e.g., super vector bundles and super connections), and in this translation the auxiliary data is a type of Chern–Simons form. We will explain this idea roughly.

In dimension 1|1, the bordisms we consider are energy zero 1|1-dimensional super paths in  $X$ , i.e., super intervals whose map to  $X$  factors through a super point. We denote the category of these under concatenation by  $\mathbf{P}_0(X)$ . Smooth functors  $F$  from  $\mathbf{P}_0(X)$  to (finite-dimensional) vector spaces are determined by super vector bundles with Quillen super connection on  $X$ . Similar to the motivating ideas in Fei Han’s thesis [Han08], the value of  $F$  on super *loops* is the differential form-valued Chern character of the super connection. An odd differential form  $\eta$  on  $X$  measures how the partition function changes as a result of “integrating out” higher energy states in the sense outlined above. Hence, we think of  $F$  as a finite-dimensional cutoff of an (unspecified) infinite-dimensional theory and  $\eta$  as an

integral over higher energy states encoding how the partition function in the cutoff theory differs from the original.

The 2|1-dimensional situation is completely analogous: consider 2|1-dimensional annuli whose map to  $X$  factors through a super point. These have energy zero in the 2|1-dimensional sigma model, and we denote the category of such by  $\mathbf{Ann}_0^{2|1}(X)$ . Smooth functors  $F$  from  $\mathbf{Ann}_0^{2|1}(X)$  to vector spaces determine a sequence of super vector bundles with super connection on  $X$ , where each term in the sequence is associated to an irreducible  $S^1$ -representation that results from the automorphism gotten by rotating annuli. After imposing an appropriate finiteness condition, we can view  $F$  as a cutoff version of a 2|1-dimensional field theory. The data of integrating out higher energy states is a sequence of odd forms, and the smooth functor  $F$  along with this extra data is what gives rise to a 2|1-dimensional low-energy annular effective field theory in our sense.

**1.5. Terminology.** Throughout,  $X$  will denote a smooth, compact, oriented manifold. Unless stated otherwise, all vector spaces and vector bundles are “super,” though sometimes we include this adjective for emphasis. Similarly, we use  $\mathrm{Tr}$  to denote the super trace of a linear map between super vector spaces and  $\otimes$  to denote the graded tensor product. We frequently use the functor of points when dealing with supermanifolds, and reserve the letter  $S$  for a test supermanifold. The isomorphism of supermanifolds,  $\pi TX \cong \mathbf{SMfld}(\mathbb{R}^{0|1}, X)$ , is used repeatedly. We refer to Appendix A for a bit more background on these (and other) ingredients.

**1.6. Acknowledgements.** Several years ago Mike Freedman explained Kitaev’s (then recent) paper to me, which planted a seed relating cohomology theories and *effective* field theories that eventually grew into this work. I also thank Ralph Cohen, Kevin Costello, Chris Douglas, Owen Gwilliam, Stephan Stolz, Vesna Stojanoska, and Peter Teichner for helpful conversations. Lastly, I am appreciative of the well-timed encouragement from Gerd Laures at the beginning of this project and from Matt Ando towards the end.

## 2. 1|1-EFFECTIVE FIELD THEORIES AND K-THEORY

The goal of this section is to prove Theorem 1.3 while also setting up definitions that readily generalize to the 2|1-dimensional case. Cocycles for differential K-theory are built from smooth functors between energy zero super paths and vector spaces. The key fact (Lemma 2.5) is that the category of these functors is equivalent to the category of super vector bundles on  $X$  with Quillen super connection. The differential form-valued Chern character can then be read off from the value on energy zero super loops in  $X$  (compare [Han08]). With these observations in place, extracting differential K-theory amounts to imposing one of the usual equivalence relations. Various approaches work, and we pick one that builds a dictionary between Chern–Simons forms and their counterparts in effective field theories.

**2.1. The Lie category of energy zero super paths.** Define an action Lie groupoid by the  $\mathbb{R}^{1|1}$ -action on  $\pi TX$  by the projection homomorphism  $\mathbb{R}^{1|1} \rightarrow \mathbb{R}^{0|1}$  composed with the  $\mathbb{R}^{0|1}$ -action on  $\pi TX$  generated by the de Rham operator (see subsection A.1). We will study the Lie subcategory

$$\mathbf{P}_0(X) := \left( \begin{array}{c} \mathbb{R}_{\geq 0}^{1|1} \times \pi TX \\ \Downarrow \\ \pi TX \end{array} \right) \subset \left( \begin{array}{c} \mathbb{R}^{1|1} \times \pi TX \\ \Downarrow \\ \pi TX \end{array} \right) = \pi TX // \mathbb{R}^{1|1}$$

gotten by restricting the action to  $\mathbb{R}_{\geq 0}^{1|1} \subset \mathbb{R}^{1|1}$ , where  $\mathbb{R}_{\geq 0}^{1|1}$  denotes the restriction of the structure sheaf of  $\mathbb{R}^{1|1}$  to  $\mathbb{R}_{\geq 0} \subset \mathbb{R}$ . The subcategory  $\mathbf{P}_0(X)$  has a geometric interpretation as a moduli space of super Euclidean intervals with a map to  $X$  that factors through the projection to the super point. Indeed, a map

$$S \rightarrow \mathrm{Ob}(\mathbf{P}_0(X)) = \pi TX \cong \mathbf{SMfld}(\mathbb{R}^{0|1}, X)$$

yields an  $S$ -family of super points  $\phi: S \times \mathbb{R}^{0|1} \rightarrow X$ . A map

$$S \rightarrow \text{Mor}(\mathbf{P}_0(X)) \cong \mathbb{R}_{\geq 0}^{1|1} \times \underline{\mathbf{SMfld}}(\mathbb{R}^{0|1}, X)$$

gives a (collared) super path in  $X$  factoring through the projection

$$(4) \quad \gamma: S \times \mathbb{R}^{1|1} \xrightarrow{\text{proj}} S \times \mathbb{R}^{0|1} \xrightarrow{\phi} X,$$

with source super point  $x_0: S \times \mathbb{R}^{0|1} \hookrightarrow S \times \mathbb{R}^{1|1} \rightarrow X$  from the standard inclusion  $\mathbb{R}^{0|1} \subset \mathbb{R}^{1|1}$  and target super point the composition

$$x_1: S \times \mathbb{R}^{0|1} \hookrightarrow S \times \mathbb{R}^{1|1} \xrightarrow{(t, \theta)} S \times \mathbb{R}^{1|1}$$

where the second map translates the fibers of  $S \times \mathbb{R}^{1|1}$  by the given  $S$ -point of  $(t, \theta) \in \mathbb{R}_{\geq 0}^{1|1}(S)$ . Such paths must have a nonnegative length, whence  $t \geq 0$ . Under this identification, concatenation of super paths corresponds to composition in  $\mathbf{P}_0(X)$ . A smooth map  $X \rightarrow Y$  induces a smooth functor  $\mathbf{P}_0(X) \rightarrow \mathbf{P}_0(Y)$  and  $\mathbf{P}_0(X)$  is local in  $X$ : it can be reconstructed from the categories  $\mathbf{P}_0(U_i)$  for  $\{U_i\}$  a cover of  $X$ .

*Remark 2.1.* For any metric on  $X$ , the paths in  $\mathbf{P}_0(X)$  have energy zero with respect to the classical super particle in  $X$ , i.e., the 1|1-dimensional sigma model; e.g., see [Fre99] Chapter 2. To explain this roughly, if we view a map  $S \times \mathbb{R}^{1|1} \rightarrow X$  as an ordinary path  $x: S \times \mathbb{R} \rightarrow X$  and a section  $\psi$  of  $\gamma^* \pi^* TX$ , the energy zero property for morphisms in  $\mathbf{P}_0(X)$  follows from  $x$  being a constant path (so it has energy zero in the usual sense of Riemannian geometry), and  $\psi$  being (covariantly) constant along  $x$ .

**Example 2.2.** The category  $\mathbf{P}_0(\text{pt})$  has one object, and its morphisms from the super semigroup  $\mathbb{R}_{\geq 0}^{1|1}$ . A representation of  $\mathbb{R}_{\geq 0}^{1|1}$  on a vector space  $V$  is determined by a single odd endomorphism  $\mathbb{A}$ :

$$(5) \quad (t, \theta) \mapsto \exp(-t\mathbb{A}^2 + \theta\mathbb{A}) \in \text{End}(V), \quad \mathbb{A} \in \text{End}(V)^{\text{odd}}$$

and this representation determines a (finite-dimensional) 1|1-Euclidean field theory in the sense of Stolz and Teichner [ST04].

**2.2. Representations of energy zero super paths.** We refer to Appendix A.3 for our definition of a smooth functor from a Lie category to  $\mathbf{Vect}$ .

**Definition 2.3.** A representation of  $\mathbf{P}_0(X)$  is a smooth functor  $\mathbf{P}_0(X) \rightarrow \mathbf{Vect}$ . Representations pull back along maps  $f: X \rightarrow Y$  via precomposition with the induced functor  $\mathbf{P}_0(X) \rightarrow \mathbf{P}_0(Y)$ , and we denote the pullback functor by  $f^*$ .

**Example 2.4.** Let  $E$  be a  $\mathbb{Z}/2$ -graded vector bundle over  $X$  equipped with a Quillen super connection  $\mathbb{A}$ . Then we claim the family of operators,

$$(6) \quad \exp(-t\mathbb{A}^2 + \theta\mathbb{A}) \in \Gamma(\mathbb{R}_{\geq 0}^{1|1} \times \pi^* TX, \text{Hom}(s^* E, t^* E))$$

defines a representation of  $\mathbf{P}_0(X)$ . To verify the formula does indeed define a morphism of vector bundles as claimed, we identify sections of  $s^* E$  and  $t^* E$  with the vector space  $C^\infty(\mathbb{R}_{\geq 0}^{1|1}) \otimes \Omega^\bullet(X; E)$  with two different module structures over the function  $C^\infty(\mathbb{R}_{\geq 0}^{1|1}) \otimes \Omega^\bullet(X)$ : we take the obvious action for  $s^* E$ , and for  $t^* E$  we take the action twisted by the map  $\Omega^\bullet(X) \rightarrow C^\infty(\mathbb{R}_{\geq 0}^{1|1}) \otimes \Omega^\bullet(X)$  given by  $\omega \mapsto \omega + \theta d\omega$ . In this description, we require that  $\exp(-t\mathbb{A}^2 + \theta\mathbb{A})$  be a linear map between the vector spaces of sections, satisfying a twisted version of linearity over functions, namely

$$e^{-t\mathbb{A}^2 + \theta\mathbb{A}}(\omega s) = (\omega + \theta d\omega) \exp(-t\mathbb{A}^2 + \theta\mathbb{A})s, \quad \omega \in \Omega^\bullet(X)$$

which follows from the Leibniz rule for  $\mathbb{A}$ . We verify compatibility with composition

$$\begin{aligned}
e^{-t\mathbb{A}^2+\theta\mathbb{A}}e^{-t'\mathbb{A}^2+\theta'\mathbb{A}} &= e^{-t\mathbb{A}^2}(1+\theta\mathbb{A})e^{-t'\mathbb{A}^2}(1+\theta'\mathbb{A}) \\
&= e^{-t\mathbb{A}^2+t'\mathbb{A}^2}(1+\theta\mathbb{A})(1+\theta'\mathbb{A}) \\
&= e^{-(t+t')\mathbb{A}^2}(1+(\theta+\theta')\mathbb{A}-\theta\theta'\mathbb{A}^2) \\
&= e^{-(t+t'+\theta\theta')\mathbb{A}^2+(\theta+\theta')\mathbb{A}}.
\end{aligned}$$

These properties show we have a representation of  $P_0(X)$ . In fact, all representations take this form.

**Lemma 2.5.** *The groupoid of representations of  $P_0(X)$  is equivalent to the groupoid whose objects are super vector bundles with Quillen super connection over  $X$ , and whose isomorphisms are isomorphisms of super vector bundles over  $\pi TX$  compatible with super connections. This equivalence is natural in  $X$ .*

*Proof of Lemma 2.5.* By trivializing along the fibers of the projection  $\pi TX \rightarrow X$ , the groupoid of super vector bundles over  $\pi TX$  is equivalent to the groupoid whose objects are super vector bundles over  $X$  and whose morphisms are isomorphisms over  $\pi TX$ , i.e., appropriately invertible elements of the set  $\Omega^\bullet(X; \text{Hom}(E, E'))$ . Via this equivalence, the data of a representation on objects is a super vector bundle over  $X$ . This is natural in  $X$  because pullback representations are defined on pulled back vector bundles.

On the morphisms of  $P_0(X)$ , a representation determines a section  $R \in \Gamma(\mathbb{R}_{\geq 0}^{1|1} \times \pi TX, \text{Hom}(s^*E, t^*E))$ . First we characterize  $R$  on its restriction to  $\mathbb{R}_{\geq 0} \times \pi TX \subset \mathbb{R}_{\geq 0}^{1|1} \times \pi TX$ . On this subspace, the source and target maps are both the projection,  $\mathbb{R}_{\geq 0} \times \pi TX \rightarrow \pi TX$ , and so the restriction of  $R$  is a section of the endomorphisms bundle. Compatibility with composition and identities requires that this section determine a family of representations of the semigroup  $\mathbb{R}_{\geq 0}$  on the fibers of  $E$  over  $\pi TX$ . By the existence and uniqueness to solutions of ordinary differential equations on supermanifolds (see [Dum06], Section 3.4) we obtain a generator  $H$ ,

$$R|_{\mathbb{R}_{\geq 0} \times \pi TX} = e^{-tH}, \quad H \in \Omega^\bullet(X, \text{End}(E)), \quad t \in \mathbb{R}_{\geq 0}.$$

Now, if we Taylor expand  $R$  in the odd variable  $\theta \in C^\infty(\mathbb{R}_{\geq 0}^{1|1})$ , we get  $R = R^{\text{ev}}(t) + \theta R^{\text{odd}}(t)$ . The locus  $\theta = 0$  is precisely  $\mathbb{R}_{\geq 0} \times \pi TX$ , and so  $R^{\text{ev}} = e^{-tH}$ , and we have

$$R = e^{-tH}(1 + \theta\mathbb{A}(t)),$$

for  $\mathbb{A}(t): s^*E \rightarrow t^*E$  an odd map between vector bundles. Compatibility with composition is the equality

$$e^{-tH}(1 + \theta\mathbb{A}(t))e^{-t'H}(1 + \theta'\mathbb{A}(t)) = e^{-(t+t'+\theta\theta')H}(1 + (\theta + \theta')\mathbb{A}(t))$$

which in turn gives  $\mathbb{A}^2 = H$  and  $\mathbb{A} = \mathbb{A}(0)$  is independent of  $t$ . Therefore the map  $\mathbb{A}$  completely determines the representation by same formula as (6).

To understand  $\mathbb{A}$ , we first view  $R$  as a map between vector spaces of sections, i.e., a linear map from  $C^\infty(\mathbb{R}_{\geq 0}^{1|1}) \otimes \Omega^\bullet(X; E)$  to itself. Since  $R$  comes from a map of vector bundles  $s^*E \rightarrow t^*E$  (i.e., modules over functions) it satisfies

$$R(\omega \cdot s) = (\omega + \theta d\omega) \cdot R(s), \quad \omega \in \Omega^\bullet(X), \quad s \in C^\infty(\mathbb{R}_{\geq 0}^{1|1}) \otimes \Omega^\bullet(X; E).$$

By definition,  $e^{-tH} = e^{-t\mathbb{A}^2}$  is linear over *all* functions (being a section of the endomorphism bundle) and so the above shows that  $\mathbb{A}$  satisfies a (graded) Leibniz rule, and hence the representation determines a super connection  $\mathbb{A}$ .  $\square$

**2.3. The renormalization group.** The *renormalization group* (RG) action on  $\mathbb{R}^{1|1}$  is

$$(t, \theta) \mapsto (\lambda^2 t, \lambda \theta), \quad (t, \theta) \in \mathbb{R}^{1|1}(S), \quad \lambda \in \mathbb{R}_{> 0}(S).$$



As we explain, this automorphism dilates the length of a super paths, inducing a family of functors  $\mathrm{RG}_\lambda: \mathbf{P}_0(X) \rightarrow \mathbf{P}_0(X)$  satisfying  $\mathrm{RG}_\lambda \circ \mathrm{RG}_\mu \cong \mathrm{RG}_{\lambda+\mu}$ . We describe these functors explicitly as  $\mathbb{R}_{>0}$ -actions on the object and morphism super manifolds of  $\mathbf{P}_0(X)$

$$(7) \quad \mathbb{R}_{>0} \times \pi TX \rightarrow \pi TX, \quad \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0}^{1|1} \times \pi TX \rightarrow \mathbb{R}_{\geq 0}^{1|1} \times \pi TX,$$

determined on  $S$ -points  $(x, \psi) \in \pi TX(S)$ ,  $(t, \theta) \in \mathbb{R}_{\geq 0}^{1|1}(S)$ , and  $\lambda \in \mathbb{R}_{>0}(S)$  by

$$(\lambda, x, \psi) \mapsto (x, \lambda^{-1}\psi), \quad (\lambda, t, \theta, x, \psi) \mapsto (\lambda^2 t, \lambda \theta, x, \lambda^{-1}\psi),$$

respectively. This pair of actions is compatible with the source, target and unit maps, so defines the functor  $\mathrm{RG}_\lambda: \mathbf{P}_0(X) \rightarrow \mathbf{P}_0(X)$  as claimed. It also dilates the super length parameters,  $(t, \theta)$ , of a super path.

Precomposing a representation with  $\mathrm{RG}_\lambda$  yields an  $\mathbb{R}_{>0}$ -action on the category of representations we also call the *RG-action*. We characterize it using Lemma 2.5. Since the  $\mathbb{R}_{>0}$ -action on  $\pi TX$  restricts to a trivial action on  $X$ , on sections  $\Omega^\bullet(X; E)$  the action is simply through the action on forms, i.e.,  $s \mapsto \lambda^{-k}s$  for  $s \in \Omega^k(X; E)$ . In terms of Equation 6, the action is

$$\begin{aligned} \exp(-t\mathbb{A}^2 + \theta\mathbb{A}) &\xrightarrow{\mathrm{RG}_\lambda} \exp(-\lambda^2 t(\mathbb{A}_0 + \lambda^{-1}\mathbb{A}_1 + \dots + \lambda^n \mathbb{A}_n)^2 \\ &\quad + \lambda \theta(\mathbb{A}_0 + \lambda^{-1}\mathbb{A}_1 + \dots + \lambda^{-n} \mathbb{A}_n)) \end{aligned}$$

which we can rephrase as an action on the space of super connections

$$\mathbb{A} \xrightarrow{\mathrm{RG}_\lambda} \lambda \mathbb{A}_0 + \mathbb{A}_1 + \lambda^{-1}\mathbb{A}_2 + \lambda^{-2}\mathbb{A}_3 + \dots \lambda^{-(n-1)}\mathbb{A}_n,$$

where  $\mathbb{A}_i$  denotes the component of  $\mathbb{A}$  with differential form degree  $i$ . This is the same rescaling action on super connections used by Bismut, Getzler, and Quillen; see [BGV92] Chapter 9.

*Remark 2.6.* To connect with the ideas of effective field theories, when  $\mathbb{A}_0$  is self-adjoint with respect to a chosen hermitian metric, the eigenvalues of  $\mathbb{A}_0^2$  on a fiber of  $E$  are the energies of a quantum-mechanical system. The RG-action varies the energy scale, as evident in the formula above. In particular, as  $\lambda \rightarrow \infty$  the RG flow sends all states with positive energy to ones with infinite energy. The limit of a representation determined by (6) on these positive energy states is the *zero* operator.

**Example 2.7.** The RG-action on a representation of  $\mathbf{P}_0(\mathrm{pt})$  is

$$\exp(-t\mathbb{A}^2 + \theta\mathbb{A}) \mapsto \exp(-t\lambda^2\mathbb{A}^2 + \theta\lambda\mathbb{A}) \in \mathrm{End}(V), \quad \lambda \in \mathbb{R}_{>0}, \quad \mathbb{A} \in \mathrm{End}(V)^{\mathrm{odd}}.$$

This action and its infinite-dimensional generalization play important roles in the index theorem by way of the McKean–Singer theorem. As  $\lambda \rightarrow \infty$  the operator  $\exp(-t\lambda^2\mathbb{A}^2)$  tends to a projection operator onto the kernel of  $\mathbb{A}$ , and the super dimension of this kernel is the index of  $\mathbb{A}$ .

**2.4. The (Chern) character of a representation.** An  $S$ -family of super loops in  $X$  is a family of super paths with positive super length  $(t, \theta) \in \mathbb{R}_{>0}^{1|1}(S) \subset \mathbb{R}_{\geq 0}^{1|1}(S)$ , whose source and target  $S$ -points are the same. Define *energy zero super loops* as the subspace

$$\mathcal{M}_0^{1|1}(X) := \mathbb{R}_{>0} \times \pi TX \subset \mathbb{R}_{\geq 0}^{1|1} \times \pi TX = \mathrm{Mor}(\mathbf{P}_0(X))$$

with  $t > 0$  and  $\theta = 0$ . For a representation of  $\mathbf{P}_0(X)$ , evaluation on an  $S$ -family of energy zero super loops yields an endomorphism of the associated vector bundle on  $S$ , and the super trace of this endomorphism gives a function on  $S$ . Using ideas similar to those in Fei Han's thesis [Han08], such functions encode the Chern character of the super connection corresponding to the representation of  $\mathbf{P}_0(X)$ . The simplest version is to restrict to the loops with circumference 1, i.e.,  $\{1\} \times \pi TX \subset \mathbb{R}_{>0} \times \pi TX = \mathcal{M}_0^{1|1}(X)$ . From Equation 6, we obtain the function

$$\mathrm{Tr}(\exp(-\mathbb{A}^2)) \in C^\infty(\pi TX) = \Omega^\bullet(X)$$

which does indeed give Quillen's version of the Chern character of a super connection. However, restriction to a particular isomorphism class of closed bordism does not generalize

nicely to the 2|1-dimensional case, wherein working over the moduli stack of tori adds considerable richness to the theory.

Instead, the rough idea is to use the RG-flow to rescale the super loops  $\mathcal{M}_0^{1|1}(X) \subset \text{Mor}(\mathbf{P}_0(X))$  so that each loop has circumference 1: take the map  $\phi_0$  into the universal family of energy zero super paths,

$$\phi_0: \mathcal{M}_0^{1|1}(X) \rightarrow \mathbb{R}_{>0} \times \mathcal{M}_0^{1|1}(X) \xrightarrow{\text{RG}} \mathcal{M}_0^{1|1}(X) \subset \text{Mor}(\mathbf{P}_0(X))$$

where the first arrow is the map,

$$\begin{aligned} (\mathcal{M}_0^{1|1}(X))(S) = (\mathbb{R}_{>0} \times \pi TX)(S) &\rightarrow (\mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \pi TX)(S) = (\mathbb{R}_{>0} \times \mathcal{M}_0^{1|1}(X))(S), \\ (t, x, \psi) &\mapsto (1/\sqrt{t}, t, x, \psi), \quad t \in \mathbb{R}_{>0}(S), \quad (x, \psi) \in \pi TX(S) \end{aligned}$$

and RG is the restriction to  $\mathcal{M}_0^{1|1}(X)$  of the RG-action (7) on morphisms in  $\mathbf{P}_0(X)$ . The map  $\phi_0$  gives an  $\mathcal{M}_0^{1|1}(X)$ -family of energy zero loops, so for a representation  $\mathbf{E}$  of  $\mathbf{P}_0(X)$ , we obtain an endomorphism  $\mathbf{E}(\phi_0)$  of a bundle over  $\mathcal{M}_0^{1|1}(X)$ .

**Definition 2.8.** The *character* of a representation  $\mathbf{E}$  of  $\mathbf{P}_0(X)$  is

$$\text{Tr}(\mathbf{E}(\phi_0)) \in C^\infty(\mathcal{M}_0^{1|1}(X)).$$

By the the cyclic property of the super trace, the character is an invariant of the isomorphism class of a representation.

Applying Lemma 2.5, the character can be computed as

$$(8) \quad \text{Tr} \left( \exp \left( (\mathbb{A}_0 + t^{1/2} \mathbb{A}_1 + \cdots + t^{n/2} \mathbb{A}_n)^2 \right) \right).$$

This function is in the subalgebra in the image of

$$(9) \quad \begin{aligned} C^\infty(\pi TX) &\hookrightarrow C^\infty(\mathcal{M}_0^{1|1}(X)) = C^\infty(\mathbb{R}_{>0} \times \pi TX) \\ \omega &\mapsto t^{k/2} \otimes \omega, \quad \omega \in \Omega^k(X) \subset C^\infty(\pi TX) \end{aligned}$$

Since the  $\mathbb{R}_{>0}$ -dependence of the character is completely determined by the degrees of its constituent forms, this allows us to identify it with a differential form.

**Notation 2.9.** Let  $Z(\mathbf{E}) \in \Omega_{\text{cl}}^{\text{ev}}(X)$  denote the closed, even differential form determined by the character of the representation  $\mathbf{E}$ .

Hence,  $Z(\mathbf{E})$  gives a differential form representative for the Chern character of the super vector bundle with super connection  $(E, \mathbb{A})$  underlying the representation  $\mathbf{E}$ .

*Remark 2.10.* That the character of a representation is a *closed* and *even* form can be understood as invariance of the function on  $\mathcal{M}_0^{1|1}(X)$  under the action of the super Euclidean isometries of super circles; loop rotation is generated by the de Rham operator so leads to closedness, and the  $\mathbb{Z}/2$  automorphism from the map  $\mathbb{R}^{1|1}/\mathbb{Z} \rightarrow \mathbb{R}^{1|1}/\mathbb{Z}$  induced by  $(s, \eta) \mapsto (s, -\eta)$  leads to evenness. For a future remark, we also observe that submersions  $X \rightarrow Y$  whose fibers are compact and oriented have a volume form along the fibers of the induced map

$$\mathcal{M}_0^{1|1}(X) \rightarrow \mathcal{M}_0^{1|1}(Y),$$

that multiplies a function by  $t^{-(\dim(X) - \dim(Y))/2}$  and integrates the differential form component along the fibers of the map. Note that the factor of  $t$  leads to an integration map that preserves the subalgebra determined by (9).

## 2.5. Chern–Simons forms and integration over higher energy states.

**Definition 2.11.** A *concordance* between a pair of representations  $\mathbf{E}_0, \mathbf{E}_1$  of  $\mathbf{P}_0(X)$  is a representation  $\tilde{\mathbf{E}}$  of  $\mathbf{P}_0(X \times \mathbb{R})$  and isomorphisms  $i_0^* \tilde{\mathbf{E}} \cong \mathbf{E}_0, i_1^* \tilde{\mathbf{E}} \cong \mathbf{E}_1$  where  $i_0, i_1: X \hookrightarrow X \times \mathbb{R}$  are the inclusions at  $X \times \{0\}$  and  $X \times \{1\}$ . If there is a concordance between  $\mathbf{E}_0$  and  $\mathbf{E}_1$  we call the pair of representations *concordant*.

**Example 2.12.** Let  $E_0$  and  $E_1$  be representations of  $P_0(X)$  corresponding to super vector bundles with super connection  $(E_0, \mathbb{A}^{E_0})$  and  $(E_1, \mathbb{A}^{E_1})$ . Then  $E_0$  and  $E_1$  are concordant if and only if there is an isomorphism,  $\phi: E_0 \rightarrow E_1$ . A concordance can be specified by a choice of path  $\mathbb{A}(\lambda)$  in the (affine) space of super connections with  $\mathbb{A}(0) = \mathbb{A}^{E_0}$  and  $\mathbb{A}(1) = \phi^* \mathbb{A}^{E_1}$ . Explicitly, this concordance is the representation determined by the vector bundle  $p^* E_0$  for  $p: X \times \mathbb{R} \rightarrow X$  the projection and super connection  $(d\lambda)\partial/\partial\lambda + \mathbb{A}(\lambda)$ . Finally, we use the isomorphism  $E_1 \cong E_0 \cong i_1^*(p^* E_0)$  to fix the claimed target of this concordance.

**Example 2.13** (The RG-flow as a concordance). The  $\mathbb{R}_{>0}$ -family of super connections  $\text{RG}_\lambda(\mathbb{A})$  on  $E \rightarrow X$  can be lifted to a concordance parametrized by  $\mathbb{R}_{>0} \cong \mathbb{R}$ .

Concordant representations have concordant characters, and the difference is measured by a *Chern–Simons form*.

**Definition 2.14.** For concordant representations  $E_0, E_1: P_0(X) \rightarrow \text{Vect}$ , define

$$\text{CS}(E_0, E_1) = \int_{X \times I/X} \in Z(\tilde{E}) \in \Omega_{\text{cl}}^{\text{odd}}(X)$$

where  $I = [0, 1]$ , and the integral is fiberwise along the projection  $X \times I \rightarrow X$ . Two choices of concordance give a pair of Chern–Simons forms that differ by a  $d$ -exact function on  $\mathbb{R}_{>0} \times \pi TX$ , so we obtain a well-defined equivalence class

$$\text{CS}(E_0, E_1) \in \Omega^{\text{odd}}(X)/d\Omega^{\text{ev}}(X)$$

without specifying a concordance.

*Remark 2.15.* The discussion in Section 2.4 lets us rephrase Chern–Simons forms in terms of functions on  $\mathcal{M}_0^{1|1}(X)$ : the de Rham operator is the vector field generated by rotating super loops, and the integral arises from the volume form on the fibers  $\mathcal{M}_0^{1|1}(X \times I) \rightarrow \mathcal{M}_0^{1|1}(X)$ . In this description,  $\text{CS}(E_0, E_1)$  determines an equivalence class of functions on  $\mathcal{M}_0^{1|1}(X)$  that measures the difference between characters of the representations. For clarity, we have elected to phrase this structure in terms of the more standard objects.

The Chern–Simons form associated to the RG-flow measures how the partition function changes as we vary the energy scale (in the sense of Remark 2.6), with the  $\lambda \rightarrow \infty$  limit (if it exists) measuring the effect of integrating out the higher energy states entirely. In the example below we describe a class of representations for which the Chern–Simons form is identically zero for all  $\lambda$ , so in particular, the limit  $\lambda \rightarrow \infty$  is well-defined.

**Example 2.16.** Let  $(V, \nabla)$  be an ordinary (purely even) vector bundle with connection on  $X$ . Consider the representation of  $P_0(X)$  with underlying vector bundle  $E = V \oplus \pi V$ , and super connection with  $\mathbb{A}_1 := \nabla \oplus \nabla$  and  $\mathbb{A}_0$  the odd endomorphism of  $E$  inherited from the identity map on  $V$ . The concordance between the super connections  $\mathbb{A}_1 = \lim_{\lambda \rightarrow 0} \text{RG}_\lambda(\mathbb{A})$  and  $\text{RG}_\lambda(\mathbb{A})$  has as its Chern–Simons form

$$\begin{aligned} \text{CS}(\mathbb{A}_1, \text{RG}_\lambda(\mathbb{A})) &= \int_0^\lambda \text{Tr} \left( \frac{d\mathbb{A}(\lambda)}{d\lambda} e^{-\mathbb{A}(\lambda)^2} \right) d\lambda \\ &= \int_0^\lambda \text{Tr} \left( \mathbb{A}_0 e^{-(\lambda^2 \mathbb{A}_0^2 + \lambda[\mathbb{A}_0, \mathbb{A}_1] + \mathbb{A}_1^2)} \right) d\lambda \\ &= \int_0^\lambda \text{Tr} \left( \mathbb{A}_0 e^{-(\lambda^2 \mathbb{A}_0^2 + \mathbb{A}_1^2)} \right) d\lambda = 0 \end{aligned}$$

since  $[\mathbb{A}_1, \mathbb{A}_0] = 0$  and the super trace of an odd endomorphism is zero. Hence,

$$\lim_{\lambda \rightarrow \infty} \text{CS}(\mathbb{A}_1, \text{RG}_\lambda(\mathbb{A})) = 0.$$

For  $t > 0$ , the  $\lambda \rightarrow \infty$  limiting operator associated to the representation (as in Equation (6)) is the *zero* operator. Hence representations of the above sort are “trivial” in that they can be deformed to the zero representation without altering the partition function. The inverse to this procedure is stabilization by representations of  $P_0(X)$  of the above form, so

an appropriate category of effective field theories should incorporate this version of stable isomorphism. This is an idea we learned from [Kit09].

**Definition 2.17.** Let  $\epsilon_V$  denote the representation of  $P_0(X)$  in Example 2.16 determined by  $E = V \oplus \pi V$  and  $\mathbb{A} = \nabla \oplus \nabla$ , and call it the *stably trivial representation* determined by  $(V, \nabla)$ .

*Remark 2.18.* With some additional properties on  $\mathbb{A}_0$  (namely, self-adjointness with respect to a metric) one can prove that the limiting Chern–Simons form converges when  $\mathbb{A}_0$  is invertible. This leads to a larger class of stably trivial representations.

**2.6. Low-energy effective field theories.** With the above preamble in place, there are several options for extracting differential K-theory from a suitable category of effective field theories. Our choices make for an easy proof. However, there are many other variations corresponding to various models for differential K-theory.

**Definition 2.19.** Define the groupoid of *1|1-dimensional, low-energy effective field theories* as having objects  $(E, \alpha)$  for  $E$  a representation of  $P_0(X)$  and an equivalence class  $\alpha \in \Omega^{\text{odd}}(X)/d\Omega^{\text{ev}}(X)$ . An *isomorphism*  $(E_0, \alpha_0) \rightarrow (E_1, \alpha_1)$  between objects is a concordance between representations  $E_0$  and  $E_1$  such that  $\alpha_0 = \alpha_1 + \text{CS}(E_0, E_1)$ .

We will often drop the adjectives *1|1-dimensional* and *low-energy* when they are implied by context. The operations of  $\oplus$  and  $\otimes$  on **Vect** give us operations on representation of  $P_0(X)$ . We extend these to effective field theories via

$$(10) \quad \begin{aligned} (E_0, \alpha_0) \oplus (E_1, \alpha_1) &= (E_0 \oplus E_1, \alpha_0 + \alpha_1), \\ (E_0, \alpha_0) \otimes (E_1, \alpha_1) &= (E_0 \otimes E_1, \alpha_0 \cdot Z(E_1) + Z(E_0) \cdot \alpha_1 + \alpha_0 \cdot d\alpha_1). \end{aligned}$$

**Definition 2.20.** A *stable isomorphism* between effective field theories  $(E_0, \alpha_0)$  and  $(E_1, \alpha_1)$  is an isomorphism of effective field theories,  $(E_0 \oplus \epsilon_V, \alpha_0) \cong (E_1, \alpha_1)$ . This generates an equivalence relation  $\sim$ , and when two effective field theories are in the same equivalence class for  $\sim$  we call them *stably equivalent*.

To unpack the above, an effective field theory is stably trivial when its representation  $E$  of  $P_0(X)$  is concordant to a representation of the form  $(V \oplus \pi V, \nabla \oplus \nabla)$  with Chern–Simons form for this isomorphism  $-\alpha$ . In particular, if  $E$  is associated to a super connection whose degree zero part is invertible, then such a concordance exists (since the even and odd subbundles are isomorphic), and then triviality requires a condition on the Chern–Simons form of this concordance.

The equivalence class  $\alpha$  leads to a modification of the character of a representation that we call the *partition function*.

**Definition 2.21.** The *partition function* of a 1|1-dimensional, low-energy effective field theory  $(E, \alpha)$  is  $Z(E, \alpha) := Z(E) + d\alpha \in \Omega_{\text{cl}}^{\text{ev}}(X)$ .

Under the inclusion  $\Omega_{\text{cl}}^{\text{ev}}(X) \hookrightarrow C^\infty(\mathcal{M}^{1|1}(X))$  from (9), we can view the partition function as a function on a moduli space of energy zero super loops over  $X$ . Stable isomorphism classes of effective field theories over  $X$  form a commutative monoid induced by  $\oplus$ ; this is our claimed model for differential K-theory.

## 2.7. Proof of Theorem 1.3.

*Proof of Theorem 1.3.* Let  $(\mathcal{V}(X), \oplus)$  be the monoidal groupoid defining the Klonoff model for differential K-theory (see §A.6). It has objects  $(E, \mathbb{A}, \alpha)$  for  $E$  a super vector bundle,  $\mathbb{A}$  a super connection, and  $\alpha \in \Omega^{\text{odd}}(X)/d\Omega^{\text{ev}}(X)$  an equivalence class of forms. Lemma 2.5 gives an evident symmetric monoidal functor from low-energy effective field theories to  $(\mathcal{V}(X), \oplus)$  and this functor induces a bijection on isomorphism classes of objects. Furthermore, the isomorphism class of the effective field theory  $(\epsilon_V, 0)$  is sent to  $(V \oplus \pi V, \nabla \oplus \nabla, 0)$  in  $\mathcal{V}(X)$ . Hence, stable equivalence classes of low-energy effective field theories induced by Definition 2.20 agrees with the equivalence relation on  $\mathcal{V}(X)$  that gives

differential K-theory as an abelian group. Furthermore, the formulas for the ring structure on isomorphism classes are identical, so we in fact have demonstrated an isomorphism of rings.  $\square$

We observe that the partition function of an effective field theory agrees with the curvature map of the associated differential K-theory class.

### 3. 2|1-EFFECTIVE ANNULAR FIELD THEORIES AND TATE K-THEORY

This section proceeds in complete analogy to the previous one, where we replace super paths by super annuli. The new feature is an  $S^1$ -automorphism group from rotating the annuli. The associated weight spaces for this action supply an additional  $\mathbb{Z}$ -grading, which can be interpreted as a sequence of vector bundles and therefore a cocycle for Tate K-theory.

**3.1. The Lie category of energy zero super annuli.** For  $r \in \mathbb{R}_{>0}(S)$ , define an  $r\mathbb{Z}$ -action on  $S \times \mathbb{R}^{2|1}$  generated by  $(z, \bar{z}, \theta) \mapsto (z + r, \bar{z} + r, \theta)$ , where  $(z, \bar{z}, \theta) \in \mathbb{R}^{2|1}(S)$  is an  $S$ -point (see Appendix A.2). On the quotient we obtain a bundle of groups with multiplication determined by the formula

$$(z, \bar{z}, \theta) \cdot (z', \bar{z}', \theta') = (z + z', \bar{z} + \bar{z}' + \theta\theta', \theta + \theta'), \quad (z, \bar{z}, \theta), (z', \bar{z}', \theta') \in \mathbb{R}^{2|1}(S).$$

There is a morphism of bundles of groups  $S \times \mathbb{R}^{2|1}/r\mathbb{Z} \rightarrow S \times \mathbb{R}^{0|1}$  given by the map  $(z, \bar{z}, \theta) \mapsto \theta$ , and so we obtain an action of  $S \times \mathbb{R}^{2|1}/r\mathbb{Z}$  on  $\mathbf{SMfld}(\mathbb{R}^{0|1}, X)(S)$  by postcomposition with the standard  $\mathbb{R}^{0|1}$ -action on  $\mathbf{SMfld}(\mathbb{R}^{0|1}, X) \cong \pi TX$ .

In analogy to the 1|1-dimensional case, let  $\bar{\mathfrak{h}}^{2|1}$  denote the restriction of the structure sheaf of  $\mathbb{R}^{2|1}$  to  $\bar{\mathfrak{h}} \subset \mathbb{C} \cong \mathbb{R}^2$ , and define the Lie subcategory

$$\mathbf{Ann}_0^{2|1}(X) := \left( \begin{array}{c} (\bar{\mathfrak{h}}^{2|1} \times \mathbb{R}_{>0})/\mathbb{Z} \times \pi TX \\ \Downarrow \\ \mathbb{R}_{>0} \times \pi TX \end{array} \right) \subset \left( \begin{array}{c} (\mathbb{R}^{2|1} \times \mathbb{R}_{>0})/\mathbb{Z} \times \pi TX \\ \Downarrow \\ \mathbb{R}_{>0} \times \pi TX \end{array} \right).$$

The groupoid on the right is almost an action groupoid:  $(\mathbb{R}^{2|1} \times \mathbb{R}_{>0})/r\mathbb{Z}$  is a bundle of groups that varies with  $r$ , and it acts through the map  $(\mathbb{R}^{2|1} \times \mathbb{R}_{>0})/r\mathbb{Z} \rightarrow \mathbb{R}^{0|1} \times \mathbb{R}_{>0}$ , where the target is a trivial bundle of groups. Below we will use the notation  $(\tau, \bar{\tau}, \theta)$  to denote an  $S$ -point of  $\bar{\mathfrak{h}}^{2|1}$ .

The subcategory  $\mathbf{Ann}_0^{2|1}(X)$  has a geometric interpretation as a moduli space of super annuli equipped with a map to  $X$  factoring through the super point, and composition corresponds to gluing super annuli. To an  $S$ -family of objects  $(r, \phi) \in (\mathbb{R}_{>0} \times \pi TX)(S)$  in  $\mathbf{Ann}_0^{2|1}(X)$ , we form the family of super circles  $S \times \mathbb{R}^{1|1}/r\mathbb{Z}$  and the map to  $X$  given by the composition

$$(11) \quad S \times \mathbb{R}^{1|1}/r\mathbb{Z} \xrightarrow{\text{proj}} S \times \mathbb{R}^{0|1}/r\mathbb{Z} \cong S \times \mathbb{R}^{0|1} \xrightarrow{\phi} X$$

where  $\text{proj}$  is induced by the map  $\mathbb{R}^{1|1} \rightarrow \mathbb{R}^{0|1}$  given by  $(t, \theta) \mapsto \theta$ , and (as usual) we have identified  $\phi \in \pi TX(S) \cong \mathbf{SMfld}(\mathbb{R}^{0|1}, X)(S)$ . The isomorphism in (11) follows from the fact that the  $\mathbb{Z}$ -action is trivial on  $\mathbb{R}^{0|1}$ . To an  $S$ -family of morphisms with  $r \in \mathbb{R}_{>0}(S)$ ,  $(\tau, \bar{\tau}, \theta) \in \bar{\mathfrak{h}}^{2|1}(S)$ , and  $\phi \in \mathbf{SMfld}(\mathbb{R}^{0|1}, X)(S)$ , we form the (collared) annulus

$$S \times \mathbb{R}^{1|1}/r\mathbb{Z} \xrightarrow{\text{in}} S \times \mathbb{R}^{2|1}/r\mathbb{Z} \xleftarrow{\text{out}} S \times \mathbb{R}^{1|1}/r\mathbb{Z}$$

where the  $\text{in}$  inclusion is the standard one,  $(t, \theta) \mapsto (t, t, \theta)$ , and the  $\text{out}$  inclusion is the standard inclusion post-composed with the action of  $(\tau, \bar{\tau}, \theta)$  on  $S \times \mathbb{R}^{2|1}$ . We obtain a map to  $X$  via the composition

$$S \times \mathbb{R}^{2|1}/r\mathbb{Z} \xrightarrow{\text{proj}} S \times \mathbb{R}^{0|1}/r\mathbb{Z} \cong S \times \mathbb{R}^{0|1} \xrightarrow{\phi} X$$

where  $\text{proj}$  is induced by the projection  $\mathbb{R}^{2|1} \rightarrow \mathbb{R}^{0|1}$ .

*Remark 3.1.* The maps of super annuli to  $X$  considered above have energy zero for the classical supersymmetric sigma model studied by Witten [Wit88] in his construction of the Witten genus. The reasons are similar to those in Remark 2.1: the underlying ordinary annulus maps to  $X$  along a constant map, and the odd section along this cylinder is covariantly constant.

The objects in  $\mathbf{Ann}_0^{2|1}(X)$  have interesting automorphisms coming from rotation of loops; this determines a groupoid called the *loop rotation groupoid* that is a Lie subgroupoid of  $\mathbf{Ann}_0^{2|1}(X)$ . The objects of the loop rotation groupoid are the same as  $\mathbf{Ann}_0^{2|1}(X)$ , and the morphisms comprise a subspace of the morphisms of  $\mathbf{Ann}_0^{2|1}(X)$  determined by the inclusion of  $\mathbb{R} \hookrightarrow \mathfrak{h} \subset \mathfrak{h}^{2|1}$  of the real axis into the closed upper-half plane. This gives

$$\left( \begin{array}{c} (\mathbb{R} \times \mathbb{R}_{>0})/\mathbb{Z} \times \pi TX \\ \Downarrow \\ \mathbb{R}_{>0} \times \pi TX \end{array} \right) \hookrightarrow \mathbf{Ann}_0^{2|1}(X)$$

where the  $\mathbb{Z}$ -action on  $\mathbb{R} \times \mathbb{R}_{>0}$  is  $(x, r) \mapsto (x+r, r)$ . The source and target maps are both the projection, and we observe that the inverse to a morphism determined by  $(x, r) \in \mathbb{R} \times \mathbb{R}_{>0}$  is  $(-x, r)$ .

**3.2. Representations of energy zero super annuli.** In order to make contact with modular forms, we need to study infinite-dimensional representations of  $\mathbf{Ann}_0^{2|1}(X)$ . This is related to the fact that the only modular forms with a polynomial  $q$ -expansion are constant. Fortunately, the infinite-dimensionality can be easily tamed: for a vector bundle over the objects of  $\mathbf{Ann}_0^{2|1}(X)$ , the loop groupoid determines an  $S^1$ -action on each fiber, and it suffices to consider representations of  $\mathbf{Ann}_0^{2|1}(X)$  where the weight spaces are finite-dimensional. Hence, we can extract from the infinite-dimensional bundle a direct sum (indexed by  $\mathbb{Z}$ ) of finite-dimensional subbundles. There are various definitions that lead to this outcome; we take a low-brow approach.

**Definition 3.2.** A *representation of  $\mathbf{Ann}_0^{2|1}(X)$*  is a sequence of smooth functors  $E(n): \mathbf{Ann}_0^{2|1}(X) \rightarrow \mathbf{Vect}$  for  $n \in \mathbb{Z}$  such that

- (1) there exists an  $N \in \mathbb{Z}$  with  $E(n)$  the zero representation on the zero vector space for all  $n < N$ , and
- (2) the action of the loop rotation groupoid in  $\mathbf{Ann}_0^{2|1}(X)$  associated to  $E(n)$  is through the representation  $z \mapsto z^{n/r} \cdot \text{id}_{E(n)}$  of  $\mathbb{R}/r\mathbb{Z}$ .

A representation has *positive energy* if  $E(n)$  is the zero representation for  $n < 0$ . An *isomorphism* of representations of  $\mathbf{Ann}_0^{2|1}(X)$  is a sequence of natural isomorphisms of functors  $E(n) \simeq E'(n)$  for  $n \in \mathbb{Z}$ .

*Remark 3.3.* The use of the term *positive energy* is in reference to the similar concept in loop group representations.

*Remark 3.4.* A variation on the above definition takes as data a *filtration* of finite-dimensional representations of  $\mathbf{Ann}_0^{2|1}(X)$  whose associated graded is the sequence of representations in the definition above. This is arguably truer to form for effective field theories since such a filtration encodes a family of energy cutoffs. However, in the examples we know, hermitian metrics abound and projection operators for the weight spaces of the  $S^1$ -action allow for one to pass between the filtered and graded objects.

**Lemma 3.5.** *The groupoid of representations of  $\mathbf{Ann}_0^{2|1}(X)$  is equivalent to the groupoid whose objects are sequences of super vector bundles  $\{E(n)\}_{n \in \mathbb{Z}}$  over  $X$  with Quillen super connections  $A(n)$  on each  $E(n)$ , and whose morphisms are isomorphisms of bundles over  $\mathbb{R}_{>0} \times \pi TX$  compatible with the super connections.*

*Proof.* By definition, a representation is determined by the data of a sequence of vector bundles  $E(n)$  on  $\mathbb{R}_{>0} \times \pi TX$  and sections  $R_n$  of  $\text{Hom}(s^*E(n), t^*E(n))$  compatible with composition. Isomorphisms between representations are sequences of isomorphisms of vector

bundles  $E(n)$  compatible with the  $R_n$ . By trivializing  $E(n)$  along the fibers of the projection  $\mathbb{R}_{>0} \times \pi TX \rightarrow \pi TX \rightarrow X$ , a representation defines vector bundles  $E(n)$  over  $X$ , and an isomorphism between representations determines an isomorphism between the pullbacks of these bundles to  $\mathbb{R}_{>0} \times \pi TX$ .

In analyzing a representation over  $\text{Mor}(\text{Ann}_0^{2|1}(X))$ , first we focus on the subcategory whose morphisms are  $(\mathbb{R}_{>0} \times \bar{\mathfrak{h}})/\mathbb{Z} \times \pi TX \subset (\mathbb{R}_{>0} \times \bar{\mathfrak{h}}^{2|1})/\mathbb{Z} \times \pi TX$ . Since the source and target maps are equal on this subcategory (namely, the projections) the restriction of a representation yields a family of bundle endomorphisms. Fixing the usual coordinates  $x$  and  $y$  on  $\bar{\mathfrak{h}}$  with  $\tau = x + iy$  and  $\bar{\tau} = x - iy$ , we denote these endomorphisms by  $R_n(x, y) \in \Gamma(\mathbb{R}_{>0} \times \pi TX; \text{End}(E(n)))$ , and we have derivatives

$$(\partial_x R_n)(0, 0) = 2\pi i n / r, \quad (\partial_y R_n)(0, 0) = -2\pi H(n),$$

where the first equation follows from the assumed action of loop rotation in the definition of a representation, and  $H(n) \in \Gamma(\mathbb{R}_{>0} \times \pi TX; \text{End}(E(n)))$  is a vector bundle endomorphism (the factor of  $-2\pi$  is for later convenience). Since

$$R_n(x, y) \circ R_n(x', y') = R_n(x + x', y + y') \in \Gamma(\mathbb{R}_{\geq 0} \times \pi TX; \text{End}(E(n)))$$

and  $R_n(0, 0) = \text{id}$ , the existence and uniqueness of solutions to ordinary differential equations over supermanifolds (see [Dum06], Section 3.4) yields

$$R_n(x, y) = e^{2\pi i n x / r} e^{-2\pi y H(n)} \in \Gamma(\mathbb{R}_{>0} \times \pi TX; \text{End}(E(n))).$$

Writing  $x = \frac{1}{2}(\tau + \bar{\tau})$  and  $y = \frac{1}{2i}(\tau - \bar{\tau})$ , we have

$$R_n(\tau, \bar{\tau}) = e^{\pi i n (\tau + \bar{\tau}) / r} e^{\pi i (\tau - \bar{\tau}) H(n)} = e^{2\pi i \tau (H(n)/2 + n/2r)} e^{-2\pi i \bar{\tau} (H(n)/2 - n/2r)} = q^{L_0^n} \bar{q}^{\bar{L}_0^n}$$

where we have adopted the standard notation  $q = e^{2\pi i \tau}$  and  $\bar{q} = e^{-2\pi i \bar{\tau}}$  and

$$L_0^n := \frac{1}{2}(H(n) + n/r), \quad \bar{L}_0^n := \frac{1}{2}(H(n) - n/r) \in \Gamma(\mathbb{R}_{>0} \times \pi TX; \text{End}(E(n))).$$

Now we analyze the general family of sections over  $(\mathbb{R}_{>0} \times \bar{\mathfrak{h}}^{2|1})/\mathbb{Z} \times \pi TX$ . Taylor expanding in  $\theta$ , this has the form

$$R_n(\tau, \bar{\tau}, \theta) = R_n^{\text{ev}}(\tau, \bar{\tau})(n) + \theta R_n^{\text{odd}}(\tau, \bar{\tau}),$$

Since the subspace  $(\mathbb{R}_{>0} \times \bar{\mathfrak{h}})/\mathbb{Z} \times \pi TX \subset (\mathbb{R}_{>0} \times \bar{\mathfrak{h}}^{2|1})/\mathbb{Z} \times \pi TX$  is precisely the locus  $\theta = 0$ , we obtain  $R_n^{\text{ev}}(\tau, \bar{\tau}) = q^{L_0^n} \bar{q}^{\bar{L}_0^n}$  and so

$$R_n(\tau, \bar{\tau}, \theta) = q^{L_0^n} \bar{q}^{\bar{L}_0^n} (1 + \theta \mathbb{A}(n))$$

for a morphism of vector bundles  $\mathbb{A}(n): \mathfrak{s}^* E(n) \rightarrow \mathfrak{t}^* E(n)$ . Viewing this as a morphism between the associated modules over  $C^\infty(\bar{\mathfrak{h}}^{2|1}) \otimes \Omega^\bullet(X)$ , we have

$$q^{L_0^n} \bar{q}^{\bar{L}_0^n} (1 + \theta \mathbb{A}(n)) (\omega \cdot s) = (\omega + \theta d\omega) \cdot q^{L_0^n} \bar{q}^{\bar{L}_0^n} (1 + \theta \mathbb{A}(n)) (s),$$

where  $s \in \Gamma(\mathbb{R}_{>0} \times \pi TX; E(n))$  and  $\omega \in C^\infty(\mathbb{R}_{>0} \times \pi TX)$ . From this we deduce that  $\mathbb{A}(n)$  defines a super connection on  $E(n)$ . Compatibility with composition demands

$$q_1^{L_0^n} \bar{q}_1^{\bar{L}_0^n} (1 + \theta_1 \mathbb{A}(n)) q_2^{L_0^n} \bar{q}_2^{\bar{L}_0^n} (1 + \theta_2 \mathbb{A}(n)) = (q_1 q_2)^{L_0^n} (\bar{q}_1 \bar{q}_2)^{\bar{L}_0^n} e^{-2\pi i \theta_1 \theta_2 \bar{L}_0^n} (1 + (\theta_1 + \theta_2) \mathbb{A}(n)),$$

and we have  $\bar{L}_0^n = \mathbb{A}(n)^2 / 2\pi$ . Hence, a sequence of super connections determines a representation.  $\square$

Using the notation as in the above proof, from  $L_0^n - \bar{L}_0^n = n/r$  we get

$$\begin{aligned} q^{L_0^n} \bar{q}^{\bar{L}_0^n} (1 + \theta \mathbb{A}(n)) &= e^{2\pi i \tau (L_0^n + n/r - n/r)} e^{-2\pi i \bar{\tau} \bar{L}_0^n} (1 + \theta \mathbb{A}) \\ &= e^{2\pi i \tau n / r} e^{2\pi i (\tau - \bar{\tau}) \bar{L}_0^n} (1 + \theta \mathbb{A}) \\ &= q^{n/r} e^{-4\pi i \text{im}(\tau) \bar{L}_0^n} (1 + \theta \mathbb{A}) \\ (12) \quad &= q^{n/r} e^{-2i \text{im}(\tau) \mathbb{A}(n)^2 + \theta \mathbb{A}(n)}. \end{aligned}$$

This formula expressing the data of a representation will be useful below.

**3.3. The renormalization group.** The *renormalization group* action on  $\mathbb{R}^{2|1}$  is

$$(z, \bar{z}, \theta) \mapsto (\lambda^2 z, \lambda^2 \bar{z}, \lambda \theta), \quad (z, \bar{z}, \theta) \in R^{2|1}(S), \quad \lambda \in \mathbb{R}_{>0}(S),$$

which induces a family of functors  $\text{RG}_\lambda: \text{Ann}_0^{2|1}(X) \rightarrow \text{Ann}_0^{2|1}(X)$  satisfying  $\text{RG}_\lambda \circ \text{RG}_\mu \cong \text{RG}_{\lambda+\mu}$ . Explicitly, we can present these functors as an  $\mathbb{R}_{>0}$ -action on the objects and morphisms of  $\text{Ann}_0^{2|1}(X)$

$$(13) \quad \begin{aligned} \mathbb{R}_{>0} \times (\mathbb{R}_{>0} \times \pi TX) &\rightarrow \mathbb{R}_{>0} \times \pi TX, \\ \mathbb{R}_{>0} \times (\mathbb{R}_{>0} \times \bar{\mathfrak{h}}^{2|1})/r\mathbb{Z} \times \pi TX &\rightarrow (\mathbb{R}_{>0} \times \bar{\mathfrak{h}}^{2|1})/r\mathbb{Z} \times \pi TX, \end{aligned}$$

determined on  $S$ -points  $\lambda \in \mathbb{R}_{>0}(S)$  (the dilation),  $r \in \mathbb{R}_{>0}(S)$  (the circumference of the object super circle),  $(\tau, \bar{\tau}, \theta) \in \bar{\mathfrak{h}}^{2|1}(S)$ , and  $(x, \psi) \in \pi TX(S)$  by

$$(\lambda, r, x, \psi) \mapsto (\lambda^2 r, x, \lambda^{-1} \psi), \quad (\lambda, r, \tau, \bar{\tau}, \theta, x, \psi) \mapsto (\lambda^2 r, \lambda^2 \tau, \lambda^2 \bar{\tau}, \lambda \theta, x, \lambda^{-1} \psi).$$

These actions are compatible with the source, target and unit maps, so define functors  $\text{RG}_\lambda$  as claimed.

Precomposing a representation of  $\text{Ann}_0^{2|1}(X)$  with  $\text{RG}_\lambda$  yields an  $\mathbb{R}_{>0}$ -action on the category of representations, which we can characterize using Lemma 3.5. Because the bundles  $E(n)$  pull back along the projections,

$$\mathbb{R}_{>0} \times \pi TX \rightarrow X,$$

and this projection is RG-invariant, the RG-action on sections of  $E(n)$  is simply determined by the action on functions on  $\mathbb{R}_{>0} \times \pi TX$ . Explicitly,

$$f(r) \otimes s \mapsto f(\lambda^2 r) \otimes \lambda^{-k} s, \quad f(r) \in C^\infty(\mathbb{R}_{>0}), \quad s \in \Omega^k(X; E(n)).$$

As for the value of a representation on super paths, the RG-action can be computed through Equation 12 as

$$\begin{aligned} e^{2\pi i \tau n/r} e^{-2\text{im}(\tau) \mathbb{A}(n)^2 + \theta \mathbb{A}(n)} &\xrightarrow{\text{RG}_\lambda} \exp(2\pi i \lambda^2 \tau / (\lambda^2 r)) \\ &\cdot \exp\left(-2\lambda^2 \text{im}(\tau) \sum_{i=0}^k \lambda^{-i} \mathbb{A}(n)_i + \lambda \theta \sum_{i=0}^k \lambda^{-i} \mathbb{A}(n)_i\right) \\ &= e^{2\pi i \tau n/r} e^{-2\text{im}(\tau) \text{RG}_\lambda(\mathbb{A}(n))^2 + \theta \text{RG}_\lambda(\mathbb{A}(n))} \end{aligned}$$

where we define

$$\text{RG}_\lambda(\mathbb{A}(n)) := \lambda \mathbb{A}(n)_0 + \mathbb{A}(n)_1 + \lambda^{-1} \mathbb{A}(n)_2 + \cdots + \lambda^{-(k-1)} \mathbb{A}(n)_k.$$

This gives a generalization of the Getzler rescaling action.

**Example 3.6.** A representation of  $\text{Ann}_0^{2|1}(\text{pt})$  is determined by a sequence of vector spaces  $E(n)$  and odd operators  $\mathbb{A}(n) \in \text{End}(E(n))^{\text{odd}}$ , with Equation 12 determining the value of a representation on annuli. The RG-action in this case is

$$q^{n/r} e^{-2\text{im}(\tau) \mathbb{A}(n)^2 + \theta \mathbb{A}(n)} \mapsto q^{n/r} e^{-2\text{im}(\tau) \lambda^2 \mathbb{A}(n)^2 + \theta \lambda \mathbb{A}(n)}$$

which we view as a 2-dimensional generalization of the rescaling action used in various aspects of the index theorem (see Example 2.7). Typically, one calls a field theory *conformal* if it is invariant under the RG-flow. Applying this idea to our setup, a representation of  $\text{Ann}_0^{2|1}(\text{pt})$  is *conformal* when  $\mathbb{A}(n) = 0$  for all  $n$ . If we start with a non-conformal representation (i.e., some  $\mathbb{A}(n) \neq 0$ ) then the limit of the RG-flow as  $\lambda \rightarrow \infty$  projects to a conformal one, analogous to how the  $\lambda \rightarrow \infty$  limit of  $e^{-\lambda^2 \not{D}^2}$  tends towards a projection onto the kernel of  $\not{D}$  in the 1|1-dimensional case. In this sense, the 2|1-dimensional analog of the kernel of  $\not{D}$  is a type of conformal field theory, namely a conformal representation of  $\text{Ann}_0^{2|1}(\text{pt})$ .



**3.4. The (Chern) character.** An  $S$ -family of super tori in  $X$  is a family of super annuli associated to an  $S$  point of  $(\mathbb{R}_{>0} \times \mathfrak{h}^{2|1})/\mathbb{Z} \subset (\mathbb{R}_{>0} \times \bar{\mathfrak{h}}^{2|1})/\mathbb{Z}$  whose source and target are the same. Define the subcategory of *energy zero super tori* as the subspace  $(\mathbb{R}_{>0} \times \mathfrak{h})/\mathbb{Z} \times \pi TX \subset (\mathbb{R}_{>0} \times \bar{\mathfrak{h}}^{2|1})/\mathbb{Z} \times \pi TX$ , i.e., energy zero super annuli with  $\text{im}(\tau) > 0$  and  $\theta = 0$ . Evaluating a representation of  $\text{Ann}_0^{2|1}(X)$  on these energy zero super tori yields an endomorphism of  $E(n)$  for each  $n \in \mathbb{Z}$ , and super traces define functions on  $(\mathbb{R}_{>0} \times \mathfrak{h})/\mathbb{Z} \times \pi TX$  for each  $n \in \mathbb{Z}$ .

As in the 1|1-dimensional case, we use the RG-flow to rescale this family of supertori parametrized by  $(\mathbb{R}_{>0} \times \mathfrak{h})/\mathbb{Z} \times \pi TX$ . Take the family of energy zero super tori over  $X$  gotten from a map  $\phi_0$  into the universal family by the composition

$$\begin{aligned} \phi_0: (\mathbb{R}_{>0} \times \mathfrak{h})/\mathbb{Z} \times \pi TX &\rightarrow \mathbb{R}_{>0} \times (\mathbb{R}_{>0} \times \mathfrak{h})/\mathbb{Z} \times \pi TX \\ &\xrightarrow{\text{RG}} (\mathbb{R}_{>0} \times \mathfrak{h})/\mathbb{Z} \times \pi TX \subset (\mathbb{R}_{>0} \times \bar{\mathfrak{h}}^{2|1})/\mathbb{Z} \times \pi TX \end{aligned}$$

where the first arrow is induced by the map,

$$(\mathbb{R}_{>0} \times \mathfrak{h})(S) \rightarrow (\mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathfrak{h})(S), \quad (r, \tau, \bar{\tau}) \mapsto (1/(2\text{im}(\tau))^{1/2}, r, (\tau, \bar{\tau}, \theta)),$$

for  $r \in \mathbb{R}_{>0}(S)$  and  $(\tau, \bar{\tau}) \in \mathfrak{h}(S)$ , and RG is the restriction to  $(\mathbb{R}_{>0} \times \mathfrak{h})/\mathbb{Z} \times \pi TX$  of the RG-action (13). We denote the source of  $\phi_0$  by  $\widetilde{\mathcal{M}}^{2|1}(X)$ ; this space classifies the moduli of energy zero tori over  $X$  that come from energy zero annuli via the map  $\phi_0: \widetilde{\mathcal{M}}^{2|1}(X) \rightarrow \text{Mor}(\text{Ann}_0^{2|1}(X))$ . For each  $n \in \mathbb{Z}$ , the map  $\phi_0$  produces a family of endomorphisms of  $E(n)$  over  $\widetilde{\mathcal{M}}^{2|1}(X)$ , denoted  $\mathbf{E}(n)(\phi_0)$ .

**Definition 3.7.** The *character* of a representation of  $\text{Ann}_0^{2|1}(X)$  is the formal sum

$$\sum_{n \in \mathbb{Z}} \text{Tr}(\mathbf{E}(n)(\phi_0))$$

of functions on  $\widetilde{\mathcal{M}}^{2|1}$ . Note the terms in this sum are zero for sufficiently negative  $n$ , and for a positive energy representation they are zero for  $n < 0$ .

We have a formula for the components of this character

$$(14) \quad \text{Tr}(\mathbf{E}(n)(\phi_0)) = q^{n/r} \cdot \text{Tr} \left( \exp \left( \left( \sum_{i=1}^k (2\text{im}(\tau))^{i/2} \mathbb{A}(n)_i \right)^2 \right) \right)$$

for each  $n \in \mathbb{Z}$ . This function lands in the subalgebra generated by

$$q^{n/r} (2\text{im}(\tau))^{i/2} \otimes \omega \in C^\infty(\widetilde{\mathcal{M}}^{2|1}(X)) \cong C^\infty((\mathbb{R}_{>0} \times \mathfrak{h})/\mathbb{Z} \times \pi TX), \quad \omega \in \Omega_{\text{cl}}^i(X).$$

Since the dependence on  $r$  and  $\bar{q}$  is completely determined by  $n$  and the degree of the form, this allows us to identify the character with a closed, even differential form with coefficients in  $\mathbb{C}[[q]][q^{-1}]$ .

**Notation 3.8.** Let

$$Z(\mathbf{E}) = \sum_{n \in \mathbb{Z}} q^n \cdot \text{Tr} \left( \exp(\mathbb{A}(n)^2) \right) \in \Omega_{\text{cl}}^{\text{ev}}(X)[[q]][q^{-1}]$$

denote the power series of forms determined by the character of a representation. The inclusions

$$(15) \quad \begin{aligned} \Omega^{\text{ev}}(X) &\hookrightarrow C^\infty(\widetilde{\mathcal{M}}_0^{2|1}(X)) \cong C^\infty((\mathbb{R} \times \bar{\mathfrak{h}})/\mathbb{Z}) \otimes C^\infty(\pi TX), \\ \omega &\mapsto q^{n/r} (2\text{im}(\tau))^{i/2} \otimes \omega, \quad \omega \in \Omega^k(X) \end{aligned}$$

for each  $n \in \mathbb{Z}$  allow us to recover the character as a formal sum of functions on  $\widetilde{\mathcal{M}}_0^{2|1}(X)$  from the power series in forms.

*Remark 3.9.* As in the 1|1-dimensional case, for submersions  $X \rightarrow Y$  with compact oriented fibers, the induced map on moduli spaces of energy zero tori,  $\widetilde{\mathcal{M}}^{2|1}(X) \rightarrow \widetilde{\mathcal{M}}^{2|1}(Y)$ , has a canonical volume form along the fibers coming from integration of forms. In order to preserve the subalgebra containing the image of characters, we modify this integration by  $(2\text{im}(\tau))^{-(\dim(X) - \dim(Y))/2}$ , which defines the desired volume form. There is also an odd

vector field on  $\widetilde{\mathcal{M}}_0^{2|1}(X)$  from tori acting on themselves by translations; as before, this is the odd derivation determined by the de Rham operator.

### 3.5. Chern–Simons forms.

**Definition 3.10.** A *concordance* between a pair of representations  $E_0, E_1$  of  $\text{Ann}_0^{2|1}(X)$  is a representation  $\tilde{E}$  of  $\text{Ann}_0^{2|1}(X \times \mathbb{R})$  and isomorphisms  $i_0^* \tilde{E} \cong E_0$ ,  $i_1^* \tilde{E} \cong E_1$  where  $i_0, i_1: X \hookrightarrow X \times \mathbb{R}$  are the inclusions at  $X \times \{0\}$  and  $X \times \{1\}$ . If there is a concordance between  $E_0$  and  $E_1$  we call the pair of representations *concordant*.

As in the 1|1-dimensional case, a pair of representations is concordant if and only if the associated vector bundles  $E_0(n)$  and  $E_1(n)$  are isomorphic for all  $n \in \mathbb{Z}$ ; a sequence of paths in the space of super connections for each  $n$  determines such a concordance. Furthermore, the difference of characters for concordant representations of  $\text{Ann}_0^{2|1}(X)$  is measured by a *Chern–Simons form*

$$Z(E_1) - Z(E_0) = d \left( \int_{X \times I/X} Z(\tilde{E}) \right) =: \text{CS}(E_1, E_0) \in \Omega^{\text{ev}}(X)[[q]][q^{-1}].$$

The integral is the usual fiberwise integration of forms along  $X \times I \rightarrow X$  for each power of  $q$ ; hence each coefficient of  $q^n$  is a standard Chern–Simons form between vector bundles with super connection. For concordant representations  $E_0$  and  $E_1$  we obtain an equivalence class

$$\text{CS}(E_1, E_0) \in (\Omega^{\text{odd}}(X)/d\Omega^{\text{ev}}(X))[[q]][q^{-1}]$$

determined by the integral above for any choice of concordance.

*Remark 3.11.* Using the map (15) and following Remark 3.9, Chern–Simons forms can be interpreted as an equivalence class of functions on  $\widetilde{\mathcal{M}}_0^{2|1}(X)$ . The integral defining the Chern–Simons form uses the volume form along the fibers of the projection  $\widetilde{\mathcal{M}}_0^{2|1}(X \times I) \rightarrow \widetilde{\mathcal{M}}_0^{2|1}(X)$  and the odd vector coming from translations of super tori described at the end of the previous subsection is the operator  $d$ .

As before, Chern–Simons forms associated to the RG-flow measure the change in the character as one changes the energy scale, and in the limit  $\lambda \rightarrow \infty$  the Chern–Simons form (if it makes sense) measures the effect of “integrating out” the subspace of states on which  $\oplus_n \mathbb{A}(n)_0$  is invertible.

**Example 3.12.** Let  $(V(n), \nabla(n))$  for  $n \in \mathbb{Z}$  be a sequence of ordinary vector bundles with connection on  $X$ , and consider the representation of  $\text{Ann}_0^{2|1}(X)$  determined by the super vector bundle  $E(n) = V(n) \oplus \pi V(n)$  and super connection with  $\mathbb{A}(n)_1 = \nabla \oplus \nabla$  and  $\mathbb{A}(n)_0$  the odd endomorphism of  $E(n)$  inherited by the identity map on  $V(n)$ . For all  $n$  we have  $[\mathbb{A}(n)_1, \mathbb{A}(n)_0] = 0$ . The family of super connections  $\lambda \mathbb{A}(n)_0 + \mathbb{A}(n)_1$  gives a concordance between  $\mathbb{A}_1$  and  $\text{RG}_\lambda(\mathbb{A})$  whose Chern–Simons form in  $\Omega_{\text{cl}}^{\text{ev}}(X)[[q]][q^{-1}]$  has coefficient of  $q^n$  determined by

$$\begin{aligned} \int_0^\lambda \text{Tr} \left( \frac{d\mathbb{A}(n)(\lambda)}{d\lambda} e^{-\mathbb{A}(n)(\lambda)^2} \right) d\lambda &= \int_0^\lambda \text{Tr} \left( \mathbb{A}(n)_0 e^{-(\lambda^2 \mathbb{A}(n)_0^2 + \lambda [\mathbb{A}(n)_0, \mathbb{A}(n)_1] + \mathbb{A}(n)_1^2)} \right) d\lambda \\ &= \int_0^\lambda \text{Tr} \left( \mathbb{A}(n)_0 e^{-(\lambda^2 \mathbb{A}(n)_0^2 + \mathbb{A}(n)_1^2)} \right) d\lambda = 0 \end{aligned}$$

where we have used in the final equality that the super trace of an odd endomorphism is zero. From this we have

$$\lim_{\lambda \rightarrow \infty} \text{CS}(\mathbb{A}_1, \text{RG}_\lambda(\mathbb{A})) = 0 \in (\Omega^{\text{odd}}(X)/d\Omega^{\text{ev}}(X))[[q]][q^{-1}]$$

On annuli with  $\text{im}(\tau) > 0$ , this infinite limit of the RG-flow tends towards the zero representation on  $E(n)$  for all  $n$ , so we can think of these sorts of representations as ones that can be deformed to zero and have an exact Chern–Simons form for this deformation. The inverse to this process is stabilization by cocycles of the form of the above example.

**Definition 3.13.** Let  $\epsilon_V$  denote the representation of  $\text{Ann}_0^{2|1}(X)$  in Example 3.12 determined by  $E(n) = V(n) \oplus \pi V(n)$  and  $\mathbb{A}(n) = \nabla(n) \oplus \nabla(n)$  by  $\epsilon_V$ , and call it a *stably trivial representation*.

### 3.6. Low-energy effective field theories.

**Definition 3.14.** Define the groupoid of 2|1-dimensional, low-energy effective annular field theories as having objects  $(E, \alpha)$  for  $E$  a representation of  $\text{Ann}_0^{2|1}(X)$  and an equivalence class  $\alpha \in (\Omega^{\text{odd}}(X)/d\Omega^{\text{ev}}(X))[[q]][q^{-1}]$ . An *isomorphism*  $(E_0, \alpha_0) \rightarrow (E_1, \alpha_1)$  between objects is a concordance between representations  $E_0$  and  $E_1$  such that  $\alpha_0 = \alpha_1 + \text{CS}(E_0, E_1)$ . A low-energy effective annular field theory is of *positive energy* if its associated representation of  $\text{Ann}_0^{2|1}(X)$  is of positive energy.

We will often drop the adjectives 2|1-dimensional and low-energy when they are implied by context. The operation  $\oplus$  on **Vect** gives an operation on representations of  $\text{Ann}_0^{2|1}(X)$ . We extend this to effective field theories via

$$(16) \quad (E_0, \alpha_0) \oplus (E_1, \alpha_1) = (E_0 \oplus E_1, \alpha_0 + \alpha_1),$$

**Definition 3.15.** A pair of effective field theories  $(E_0, \alpha_0)$  and  $(E_1, \alpha_1)$  are *stably isomorphic* if there is an isomorphism of effective field theories  $(E_0 \oplus \epsilon_V, \alpha_0) \cong (E_1, \alpha_1)$ . This generates an equivalence relation on the category of effective field theories over  $M$ . When two effective field theories are in the same equivalence class, we call them *stably equivalent*.

To unpack the above, an effective field theory  $(E, \alpha)$  is stably trivial when its representation  $E$  of  $\text{Ann}_0^{2|1}(X)$  is concordant to a representation as in Example 3.12 with Chern–Simons form  $-\alpha$ . Note that if  $E$  is associated to a sequence of super connections whose degree zero part is invertible, then such a concordance exists (since the even and odd subbundles are isomorphic), and then triviality requires a condition on the Chern–Simons forms of this concordance.

**Definition 3.16.** The *partition function* of a 2|1-dimensional, low-energy effective field theory  $(E, \alpha)$  is  $Z(E, \alpha) := Z(E) + d\alpha \in \Omega_{\text{cl}}^{\text{ev}}(X)[[q]][q^{-1}]$ .

The ring structure on 2|1-dimensional effective field theories combines the ring structure on representations with  $\mathbb{Z}[[q]][q^{-1}]$ . Namely, for effective field theories representation  $(E, \alpha)$  and  $(E', \alpha')$ , define the  $n^{\text{th}}$  component of the product as

$$[\bigoplus_{i+j=n} E(i) \otimes E(j)', \sum_{i+j=n} \alpha_i \wedge Z(E(j)') + Z(E(i)) \wedge \alpha'_j + \alpha_i \wedge d\alpha'_j].$$

*Proof of Theorem 1.1.* For each  $n \in \mathbb{Z}$ , a 2|1-dimensional effective annular field theory defines a triple  $(E(n), \mathbb{A}(n), \alpha(n))$  that we can view as a cocycle in differential K-theory as in the proof of Theorem 1.3. Furthermore, isomorphisms of 2|1-dimensional field theories correspond to sequences of isomorphisms between these cocycles in the groupoid  $\mathcal{V}(X)$  of §A.6. Similarly, stable isomorphisms of 2|1-dimensional effective field theories correspond to sequences of stable isomorphisms between cocycles. This gives a well-defined map from equivalence classes of 2|1-dimensional effective field theories to  $\widehat{K}_{\text{Tate}}(X)$ . We obtain in inverse map that sends a sequence of cocycles  $(E(n), \mathbb{A}(n), \alpha(n))$  to the associated 2|1-dimensional effective field theory using Equation 12, and hence we have the desired isomorphism between abelian groups. Given our definition of the ring structure, it immediately follows that we in fact obtain an isomorphism of rings.  $\square$

As was the case for ordinary K-theory, the partition function of an effective annular field theory is the curvature of the associated class in  $\widehat{K}_{\text{Tate}}(X)$ .

**Corollary 3.17.** *There is a natural isomorphism of rings,*

$$\widehat{K}(X)[[q]] \cong 2|1\text{-AFT}_{\text{eff}}^+(X)/\sim$$

*with stable isomorphism classes of positive energy 2|1-dimensional effective annular field theories over  $X$ .*

*Proof.* Restricting to positive energy representations of super annuli produces classes in  $\widehat{K}[[q]] \subset \widehat{K}[[q]][q^{-1}]$ , and the result is immediate.  $\square$

#### 4. DEGREE $n$ EFFECTIVE FIELD THEORIES AND KMF

In the previous section we showed that characters of representations of  $\text{Ann}_0^{2|1}(X)$  define formal power series of functions on the moduli space  $\widehat{\mathcal{M}}_0^{2|1}(X)$  that classifies tori with a chosen meridian. This meridian super circle corresponds to the source and target of a morphism in  $\text{Ann}_0^{2|1}(X)$ . To promote a representation of  $\text{Ann}_0^{2|1}(X)$  to any reasonable notion of field theory, this formal sum of functions on  $\widehat{\mathcal{M}}_0^{2|1}(X)$  must (1) converge to an honest function, and (2) it cannot depend on the chosen meridian, i.e., it must pullback from the moduli stack of tori without such additional data. More generally, if the representation of  $\text{Ann}_0^{2|1}(X)$  is related to a *twisted* or *anomalous* field theory we could ask that the function on  $\widehat{\mathcal{M}}_0^{2|1}(X)$  pull back from a section of a line bundle over the moduli stack of tori without any choice of meridian.

Viewing modular forms as sections of a line bundles over the moduli stack of elliptic curves, it is tempting to define a degree  $n$  representation of  $\text{Ann}_0^{2|1}(X)$  as one whose partition function defines a degree  $n$  modular form in this obvious sense. However, 2|1-dimensional field theories come equipped with their own version of degree from the  *$n$ -free fermions*. It turns out that incorporating this version both leads to modular forms and explains normalization factors by powers of the Euler  $\Phi$ -function that show up in geometric examples. Crucial to this discussion is Stolz and Teichner's version of  $n$ -free fermions as degree  $n$  twists for their 2|1-Euclidean field theories [ST11]; we restrict their definition to energy zero super annuli.

**4.1. An aside on free fermions.** The  $n$ -free fermions are a 2|1-dimensional generalization of the  $n^{\text{th}}$  Clifford algebra. To explain the analogy, we consider functors

$$\mathbb{C}l(n): \mathbf{P}_0(\text{pt}) \rightarrow \mathbf{Alg}$$

where  $\mathbf{Alg}$  is the bicategory<sup>2</sup> of algebras, bimodules and intertwiners, and we have promoted  $\mathbf{P}_0(\text{pt})$  to a bicategory whose objects and 1-morphisms are as before, and whose 2-morphisms are Euclidean isometries of super paths (which in our description is just a  $\mathbb{Z}/2$  induced by  $(t, \theta) \mapsto (t, -\theta)$  on the path). The value of the functor  $\mathbb{C}l(n)$  on the point is the  $n^{\text{th}}$  Clifford algebra, on a super path we take the identity bimodule, and to the  $\mathbb{Z}/2$  isometry on super paths we apply the parity involution on the bimodule. Then a  $\mathbb{C}l(n)$ -twisted representation of  $\mathbf{P}_0(X)$  is a natural transformation

$$\begin{array}{ccc} & \mathbb{C}l(n) & \\ & \downarrow \text{F} & \\ \mathbf{P}_0(X) & \xrightarrow{\quad \quad} & \mathbf{Alg}, \\ & \uparrow \mathbb{1} & \end{array}$$

where  $\mathbb{1}$  is the constant functor to the unit of  $\mathbf{Alg}$ . When restricted to energy zero super circles, these twisted representations have a supertrace that takes values in a line bundle whose fiber is the super abelianization of  $\mathbb{C}l(n)$ . A mild generalization of Lemma 2.5 shows that such natural transformations  $F$  are determined by a bundle of Clifford algebras on  $X$  and a  $\mathbb{C}l(n)$ -linear super connection. In this description, the supertrace of the twisted representation matches the (Clifford) supertrace of [BGV92], Chapter 3.

*Remark 4.1.* These  $\mathbb{C}l(n)$ -twisted representations define cocycles for  $K^n(X)$  for all  $n$ . However the cocycle map isn't always surjective in odd degrees, e.g.,  $X = S^1$ .

<sup>2</sup>We'll be impressionistic in our use of higher-categorical machinery; however, all the objects we consider are simplified versions of carefully defined twisted field theories in [ST11].

Similarly, we define the  $n$ -free fermions functor

$$\mathrm{Fer}(n): \mathrm{Ann}_0^{2|1}(\mathrm{pt}) \rightarrow \mathrm{Alg}$$

whose value on objects is roughly the Clifford algebra of the free loop space  $LC^n$  with its  $\mathbb{C}$ -bilinear pairing; more precisely it is the algebra bundle on  $\mathbb{R}_{>0}$  with fiber

$$A_r := (\mathbb{C}l_1 \otimes \bigotimes_{m \in \mathbb{N}} \mathrm{Cl}(H(\mathbb{C}_m)))^{\otimes n}$$

where  $\mathbb{C}_m$  is  $\mathbb{C}$  as a vector space (we explain the subscript below) and  $\mathrm{Cl}(H(\mathbb{C}_m))$  denotes the Clifford algebra of  $\mathbb{C}_m \oplus \mathbb{C}_m^*$  equipped with the hyperbolic pairing,

$$q_H([v, w], [v', w']) = v(w') + v'(w).$$

The infinite tensor product is the *restricted* one, meaning we take the closure of finite sums of infinite products whose factors are 1 for all but finitely many  $m \in \mathbb{N}$ . Let  $(\tau, \bar{\tau}, \theta) \in \bar{\mathfrak{h}}^{2|1}(S)$  act on  $\mathbb{C}_m$  by  $e^{2\pi i m \tau / r}$ , and modify the right action of  $A_r$  on itself by the map induced by this action while keeping the standard left action. This gives a family of bimodules over  $\mathbb{R}_{>0} \times \bar{\mathfrak{h}}^{2|1}$ . As in the 1|1-dimensional case, we also have an action of parity involution on this bundle of bimodules coming from the isometry of tori induced by  $(z, \bar{z}, \theta) \mapsto (z, \bar{z}, -\theta)$ .

*Remark 4.2.* The connection between  $A_r$  and  $\mathrm{Cl}(LC^n)$  comes through Fourier expansion:  $\mathbb{C}l_1$  is the Clifford algebra of the constant loops in  $\mathbb{C}$ , and  $\mathrm{Cl}(H(\mathbb{C}_m))$  is the Clifford algebra coming from the  $m^{\mathrm{th}}$  and  $-m^{\mathrm{th}}$  Fourier modes.

Precomposition with the functor  $\mathrm{Ann}_0^{2|1}(X) \rightarrow \mathrm{Ann}_0^{2|1}(\mathrm{pt})$  gives functors we also denote by  $\mathrm{Fer}(n): \mathrm{Ann}_0^{2|1}(X) \rightarrow \mathrm{Alg}$ . A  $\mathrm{Fer}(n)$ -twisted representation of  $\mathrm{Ann}_0^{2|1}(X)$  is a natural transformation  $F$

$$\begin{array}{ccc} & \mathrm{Fer}(n) & \\ \mathrm{Ann}_0^{2|1}(X) & \begin{array}{c} \curvearrowright \\ \Downarrow F \\ \curvearrowleft \end{array} & \mathrm{Alg} \\ & \mathbb{1} & \end{array}$$

We observe that  $\mathrm{Fer}(0) \simeq \mathbb{1}$ , and  $\mathrm{Fer}(0)$ -linear representations are the same as ordinary representations of  $\mathrm{Ann}_0^{2|1}(X)$ . Twisted representations have a supertrace that takes values in a line bundle over the moduli stack of energy zero tori. As explained in [ST11] Section 6, sections of tensor powers of this line bundle are closely related to modular forms.

Now we come to the main point: the value of the  $2n^{\mathrm{th}}$  free-fermion twist is (Morita) equivalent to the  $0^{\mathrm{th}}$  twist over objects, and fixing one such equivalence will allow us to pass back and forth between twisted and untwisted representations. In more detail, we require a natural isomorphism  $F: \mathrm{Fer}(2n) \Rightarrow \mathbb{1}$ . Over the object super circles, we choose the invertible bimodule

$$M^{2n} = (M_0)^n \otimes \left( \bigotimes_{m \in \mathbb{N}} M_m \right)^{\otimes 2n},$$

where  $M_0$  is an irreducible  $\mathrm{Cl}(2)$ - $\mathbb{C}$  bimodule and the  $M_m$  are irreducible  $\mathrm{Cl}(H(\mathbb{C}_m))$ - $\mathbb{C}$  bimodules. We choose these bimodules so that the positive Fourier modes in  $LC$  act as creation operators, and negative Fourier modes as annihilators. We obtain an action by the semigroup  $\mathfrak{h}^{2|1}$  encoded through the notation  $\mathbb{C}_m$  (as above), leading to isomorphisms of bimodules over the morphism super annuli, and a natural isomorphism  $\mathrm{Fer}(2n) \Rightarrow \mathrm{Fer}(0) \simeq \mathbb{1}$ .

It remains to understand the change in the super traces over energy zero super tori under this equivalence. We claim the difference is measured by a function on  $\widetilde{M}_0^{2|1}(\mathrm{pt}) = (\mathbb{R}_{>0} \times \bar{\mathfrak{h}}^{2|1})/\mathbb{Z}$ . Our description of  $\mathrm{Fer}(2n)$  in terms of data over  $\bar{\mathfrak{h}}$  leads to a trivialization of the line bundle in which traces of twisted representations take values, and the (Morita) equivalence  $\mathrm{Fer}(2n) \Rightarrow \mathbb{1}$  gives a second trivialization. These trivializations differ by a non-vanishing function on  $(\mathbb{R}_{>0} \times \bar{\mathfrak{h}})/\mathbb{Z}$ , which we compute by analyzing how the bimodule  $M^{2n}$

alters the supertrace. Each tensor factor  $M_m$  contributes a factor of  $1 - q^{m/r}$  since the action of  $\bar{\mathfrak{h}}^{2|1}$  is trivial on the even part and is by  $q^{m/r}$  on the odd part of  $M_m$ . The tensor factor  $M_0$  contributes a factor of 1, since the  $\bar{\mathfrak{h}}$ -action on it is trivial. Therefore the overall modification is the product of these factors, yielding the  $2n^{\text{th}}$  power of

$$\Phi(q) := \prod_n (1 - q^{n/r}),$$

which is essentially the Euler  $\Phi$ -function (modulo the dependence on  $r$ ). Therefore, to identify the super trace of an untwisted representation of  $\text{Ann}_0^{2|1}(X)$  with the supertrace of a  $\text{Fer}(2n)$ -twisted representation, we must divide the function on  $\widetilde{\mathcal{M}}_0^{2|1}(X)$  by the  $2n^{\text{th}}$  power of  $\Phi(q)$ .

*Remark 4.3.* Using a related Morita-type equivalence between  $\text{Fer}(n)$  and  $\text{Cl}(n)$  and a variation of Lemma 3.5,  $\text{Fer}(n)$ -twisted representations of  $\text{Ann}_0^{2|1}(X)$  are equivalent to sequences of  $\text{Cl}(n)$ -modules with Clifford linear super connections. This gives a map to  $\text{K}_{\text{Tate}}^n(X)$  for all  $n$ , but may not be surjective when  $n$  is odd. A way to obtain *all* odd cocycles is to consider degree  $2n$  *compactly supported* cocycles on  $X \times \mathbb{R}$ . Roughly, this is an  $\mathbb{R}$ -parameter family in the category of cocycles on  $X$  that is a stably trivial cocycle “near infinity.” One drawback of this second approach is that (tensor) products of cocycles are not as easy to describe.

*Remark 4.4.* Using a related Morita-type equivalence between  $\text{Fer}(n)$  and  $\text{Cl}(n)$  and a variation of Lemma 3.5,  $\text{Fer}(n)$ -twisted representations of  $\text{Ann}_0^{2|1}(X)$  are equivalent to sequences of  $\text{Cl}(n)$ -modules with Clifford linear super connections. This gives a map to  $\text{K}_{\text{Tate}}^n(X)$  for all  $n$ , but may not be surjective when  $n$  is odd.

**4.2. Super tori over  $X$ .** An equivalent description of the stack of energy zero tori with chosen meridian (see subsection 3.4) is

$$\widetilde{\mathcal{M}}^{2|1}(X) = (\mathbb{R}_{>0} \times \mathfrak{h} \times \pi TX) // \mathbb{Z},$$

where the action is determined by the  $\mathbb{Z}$ -action on  $(\mathbb{R}_{>0} \times \mathfrak{h})$  through  $(r, \tau, \bar{\tau}) \mapsto (r, \tau + r, \bar{\tau} + r)$ . The equivalence with the previous (non-stacky) quotient description of  $\widetilde{\mathcal{M}}^{2|1}(X)$  follows because the  $\mathbb{Z}$ -action is free. Below we study the effect of forgetting the meridian.

An  $S$ -point  $(r, \tau, \bar{\tau}) \in (\mathbb{R}_{>0} \times \mathfrak{h})(S)$  determines the family of super tori  $S \times \mathbb{R}^{2|1} / (r\mathbb{Z} \oplus \tau\mathbb{Z})$  from the  $\mathbb{Z} \oplus \mathbb{Z}$ -action generated by

$$(z, \bar{z}, \theta) \mapsto (z + r, \bar{z} + r, \theta), \quad (z, \bar{z}, \theta) \mapsto (z + \tau, \bar{z} + \bar{\tau}, \theta), \quad (z, \bar{z}, \theta) \in \mathbb{R}^{2|1}(S).$$

This action commutes with the projection to  $S \times \mathbb{R}^{0|1}$ , and so an  $S$ -point of  $\pi TX$  determines a map from the  $S$ -family of super tori to  $X$  via the composition

$$S \times \mathbb{R}^{2|1} / (r\mathbb{Z} \oplus \tau\mathbb{Z}) \rightarrow S \times \mathbb{R}^{0|1} \rightarrow X.$$

Elements  $m \in \mathbb{Z}$  give automorphisms that change the lattice but preserve the meridian, i.e.,  $r$  is fixed and  $(\tau, \bar{\tau}) \mapsto (\tau + mr, \bar{\tau} + mr)$ . Forgetting the choice of meridian amounts to enlarging the automorphism group to encompass all possible changes of lattice; initially this is an extension of the  $\mathbb{Z}$ -action to an  $\text{SL}_2(\mathbb{Z})$ -action, but we will see we really need an action by the nontrivial double cover,  $\text{MP}_2(\mathbb{Z})$ .

The action of  $\text{SL}_2(\mathbb{Z})$  on the lattice generators,  $r \in \mathbb{R}_{>0}(S)$  and  $(\tau, \bar{\tau}) \in \mathfrak{h}$  is

$$(\tau, r) \mapsto (a\tau + br, c\tau + dr), \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{Z})$$

where we view both  $\mathbb{R}_{>0}$  and  $\mathfrak{h}$  as submanifolds of  $\mathbb{C}$  to make sense out of the addition. Rotating these new lattice generators by  $|c\tau + dr|/(c\tau + dr)$  returns the second generator to the subspace  $\mathbb{R}_{>0} \subset \mathbb{C}$ . However, to lift this rotation from an action on  $\mathbb{C} \cong \mathbb{R}^2$  to an action on  $\mathbb{R}^{2|1}$  compatible with the super Euclidean geometry (see §A.2) requires a *square root* of this rotation in the odd direction. This can be interpreted as (and essentially is) a lift of the rotation to a spinor bundle on the torus corresponding to the trivial square root

of the canonical bundle. A square root is determined by a choice of lift from  $\mathrm{SL}_2(\mathbb{Z})$  to an element of  $\mathrm{MP}_2(\mathbb{Z})$ , and then the resulting rotated lattice is generated by

$$r|c\tau/r + b| \in \mathbb{R}_{>0}(S), \quad \frac{a\tau/r + b}{c\tau/r + d} r|c\tau/r + d| \in \mathfrak{h}(S),$$

and the map to  $X$

$$S \times \mathbb{R}^{2|1}/(\mathbb{Z} \oplus Z) \rightarrow S \times \mathbb{R}^{0|1} \rightarrow X$$

is modified by the square root of the rotation acting on  $\mathbb{R}^{0|1}$ . The effect on  $\pi TX$  in terms of  $S$ -points is

$$(x, \psi) \mapsto \left( x, \left( \frac{|c\tau + dr|}{c\tau + dr} \right)^{1/2} \psi \right), \quad (x, \psi) \in \pi TX(S),$$

and on differential forms acts by the square root raised to the degree of the form.

**Definition 4.5.** Define the stack of *energy zero super tori over  $X$*  as

$$\mathcal{M}^{2|1}(X) := (\mathbb{R}_{>0} \times \pi TX \times \mathfrak{h})/\mathrm{MP}_2(\mathbb{Z})$$

where  $\mathrm{MP}_2(\mathbb{Z})$  acts on  $\mathbb{R}_{>0} \times \mathfrak{h}$  through the homomorphism  $\mathrm{MP}_2(\mathbb{Z}) \rightarrow \mathrm{SL}_2(\mathbb{Z})$ ,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (r, \tau, \bar{\tau}) \mapsto r|c\tau/r + b|, \frac{a\tau/r + b}{c\tau/r + d} |c\tau/r + d|r, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

and the action on  $\pi TX$  is through

$$\omega \mapsto \left( \frac{|c\tau/r + d|}{c\tau/r + d} \right)^{k/2} \omega, \quad \omega \in \Omega^k(X) \subset C^\infty(\pi TX).$$

The homomorphism of groups

$$\mathbb{Z} \rightarrow \mathrm{MP}_2(\mathbb{Z}), \quad n \mapsto \left( \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, +1 \right)$$

induces a homomorphism of Lie groupoids,

$$u: \widetilde{\mathcal{M}}^{2|1}(X) \rightarrow \mathcal{M}^{2|1}(X),$$

that associates to a “glued-up” super annulus its underlying super torus, i.e., forgets a chosen meridian. Define a line bundle  $\mathcal{L}$  on  $\mathcal{M}^{2|1}(X)$  whose sections are functions on  $\mathbb{R}_{>0} \times \mathfrak{h} \times \pi TX$  with the transformation property,

$$f(A \cdot x) = (c\tau/r + d)^{-1/2} f(x), \quad A \in \mathrm{MP}_2(\mathbb{Z}).$$

The pullback  $u^* \mathcal{L}^{\otimes 2n}$  trivializes, so we can identify its sections with functions on  $\widetilde{\mathcal{M}}^{2|1}(X)$ . In fact, our description of sections of  $\mathcal{L}$  as functions with properties specifies one such trivialization.

**Definition 4.6.** Call a function  $f \in C^\infty(\widetilde{\mathcal{M}}^{2|1}(X))$  *degree  $2n$*  if it pulls back from a section of  $\mathcal{L}^{\otimes 2n}$  over  $\mathcal{M}^{2|1}(X)$  along  $u: \widetilde{\mathcal{M}}^{2|1}(X) \rightarrow \mathcal{M}^{2|1}(X)$ .

#### 4.3. The modularity condition and $\mathrm{TMF}(X) \otimes \mathbb{C}$ .

**Definition 4.7.** A representation of  $\mathbf{E}$  of  $\mathrm{Ann}_0^{2|1}(X)$  is *trace-class* if its character  $Z(\mathbf{E}) \in \Omega_{\mathrm{cl}}^{\mathrm{ev}}(S)[[q]][q^{-1}]$  converges, i.e., is in the image of the  $q$ -expansion map

$$\Omega_{\mathrm{cl}}^{\mathrm{ev}}(X) \otimes C^\infty(\mathfrak{h}/\mathbb{Z}) \rightarrow \Omega_{\mathrm{cl}}^{\mathrm{ev}}(X)[[q]][q^{-1}].$$

**Definition 4.8.** A representation  $\mathbf{E}$  of  $\mathrm{Ann}_0^{2|1}(X)$  is *degree  $2n$*  if it is trace class and its *normalized character*  $Z(\mathbf{E})/\Phi^{2n} \in C^\infty(\widetilde{\mathcal{M}}^{2|1}(X))$  is degree  $2n$ .

The above turns out to be a rather strong condition; in the examples coming from bundles of string manifolds, the associated representations of  $\mathrm{Ann}_0^{2|1}(X)$  are rarely modular. Fortunately, effective field theories allow us to modify the character and require that only the resulting partition function be modular.

**Definition 4.9.** A 2|1-dimensional low-energy effective annular field theory  $\mathbf{E}$  has *degree*  $2n$  when its underlying representation of  $\mathbf{Ann}_0^{2|1}(X)$  is trace class and the *normalized partition function*  $Z(\mathbf{E}, \alpha)/\Phi(q)^{2n} \in C^\infty(\mathcal{M}^{2|1}(X))$  has degree  $2n$ . We call the category of these *degree*  $2n$  *effective Euclidean field theories*, and denote the category of such by  $2|1\text{-EFT}_{\text{eff}}^{2n}(X)$ .

Our next goal is to connect this story with  $\text{TMF}(X) \otimes \mathbb{C}$ . We observe that this cohomology theory can be computed as the de Rham cohomology of  $X$  with coefficients in the ring MF weak modular forms. The zeros in odd degree,  $\text{MF}^{2n+1} = \{0\}$ , imply that the cocycle map

$$\bigoplus_{i+j=n} \Omega_{\text{cl}}^{2j}(X; \text{MF}^{2i}) \rightarrow \text{TMF}^{2n}(X) \otimes \mathbb{C}$$

is surjective.

**Proposition 4.10.** *The partition function of an object in  $2|1\text{-EFT}_{\text{eff}}^{2n}(X)$  defines a de Rham cocycle representing a class in  $\text{TMF}^{2n}(X) \otimes \mathbb{C}$ .*

*Proof.* In Section 3.4 we showed that for each power of  $q$ , the components of the partition function of an effective field theory are in the image of the map

$$\Omega_{\text{cl}}^\bullet(X) \otimes \mathcal{O}(\mathfrak{h}/\mathbb{Z}) \hookrightarrow C^\infty(\mathbb{R}_{>0} \times \mathfrak{h}/\mathbb{Z} \times \pi TX), \quad \omega \otimes q^n \mapsto \omega \otimes q^{n/r} \text{im}(\tau)^j, \quad \omega \in \Omega_{\text{cl}}^{2j}(X).$$

The trace class condition guarantees that the formal sum of these components converges, so the partition function (as an honest sum) is also in the image of this map. Furthermore, its independence of  $\bar{q}$  implies holomorphicity in  $\mathfrak{h}/\mathbb{Z}$ .

From here it remains to analyze the modularity condition. Because we have restricted to the even case, the  $\text{MP}_2(\mathbb{Z})$ -action factors through  $\text{SL}_2(\mathbb{Z})$ ; this action on  $q^n \text{im}(\tau) \cdot \omega \in \Omega_{\text{cl}}^{2j}(X) \otimes C^\infty(\mathfrak{h})$  is determined by

$$\text{im}(\tau) \mapsto \frac{\text{im}(\tau/r)}{|c\tau/r + d|^2}, \quad \omega \mapsto \left( \frac{|c\tau/r + d|}{c\tau/r + d} \right)^j \omega,$$

so on  $(2\text{im}(\tau))^j \otimes \omega$  the action is through  $(c\tau/r + d)^{-j}$ . Using equation 15 to translate this into a statement for associated elements of  $\Omega_{\text{cl}}^{2j}(X) \otimes \mathcal{O}(\mathfrak{h})$ , we find the degree  $2n$  condition is the equality

$$\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \sum_k a_k q^k \right) \otimes (c\tau + d)^{-j} \omega = (c\tau + d)^{-n} \left( \sum_k a_k q^k \right) \otimes \omega,$$

and so the coefficient of the closed  $2j$ -form  $\omega$  transforms as an element of  $\text{MF}^{2(n-j)}$ . Thus, we have factored the normalized partition function of modular effective field theories as

$$2|1\text{-EFT}_{\text{eff}}^{2n}(X) \rightarrow \bigoplus_{i+j=n} \Omega_{\text{cl}}^{2j}(X) \otimes \text{MF}^{2i} \rightarrow \Omega_{\text{cl}}^{\text{ev}}(X)[[q]][q^{-1}],$$

which identifies partition functions with cocycles for classes in  $\text{TMF}^{2n}(X) \otimes \mathbb{C}$ .  $\square$

#### 4.4. Proof of Theorem 1.2.

*Proof of Theorem 1.2.* By construction, stable isomorphism classes of degree  $2n$  effective field theories over  $X$  comprise the classes in  $\widehat{\mathbf{K}}_{\text{Tate}}(X)$  whose normalized partition functions in  $\Omega_{\text{cl}}^{\text{ev}}(X)[[q]][q^{-1}]$  come from a differential form with values in modular forms, so we have a pullback square

$$\begin{array}{ccc} 2|1\text{-EFT}_{\text{eff}}^{2n}(X)/\sim & \xrightarrow{\quad} & \widehat{\mathbf{K}}(X)[[q]][q^{-1}] \cong 2|1\text{-AFT}_{\text{eff}}(X)/\sim \\ \Phi^{-2n} \cdot Z_{\text{eff}} \downarrow & & \downarrow R = Z \\ \bigoplus_{i+j=n} \Omega_{\text{cl}}^{2i}(X; \text{MF}^{2j}) & \xrightarrow{\quad \Phi \quad} & \Omega_{\text{cl}}^{\text{ev}}(X)[[q]][q^{-1}] \end{array}$$



where  $\Phi$  is the same as in equation 20. But by our definition, this shows

$$\widehat{\text{KMF}}^{2n}(X) \cong 2|1\text{-EFT}_{\text{eff}}^{2n}(X)/\sim,$$

proving the claim.  $\square$

**Corollary 4.11.** *Low-energy equivalence classes of the full subcategory of 2|1-Euclidean effective field theories over  $X$  of positive energy give a model for differential Kmf.*

*Proof.* The restriction to positive energy requires that the  $q$ -expansion of the associated modular forms have no negative powers of  $q$ , so we obtain the pullback square

$$\begin{array}{ccc} 2|1\text{-EFT}_{\text{eff}}^{2n,+}(X)/\sim & \xrightarrow{\quad} & \widehat{\text{K}}(X)[[q]] \cong 2|1\text{-AFT}_{\text{eff}}^{+}(X)/\sim \\ \Phi^{-2n} \cdot Z_{\text{eff}} \downarrow & & \downarrow R \\ \bigoplus_{i+j=n} \Omega_{\text{cl}}^i(X; \text{mf}^j) & \xrightarrow{\quad \phi \quad} & \Omega_{\text{cl}}^{\text{ev}}(X)[[q]] \end{array}$$

where  $2|1\text{-EFT}_{\text{eff}}^{2n,+}(X)$  denotes the full subcategory of  $2|1\text{-EFT}_{\text{eff}}^{2n}(X)$  whose associated representations of  $\text{Ann}_0^{2|1}(X)$  have positive energy. Hence, the claimed result follows immediately.  $\square$

By construction, the product structure on  $2|1\text{-AFT}_{\text{eff}}(X)$  is compatible with the product of effective partition functions. This gives product structures on  $2|1\text{-EFT}_{\text{eff}}^{2\bullet}(X)$  and  $2|1\text{-EFT}_{\text{eff}}^{2\bullet,+}(X)$  induced from the tensor product of representations and multiplication of differential forms.

## 5. SOME DIFFERENTIAL-GEOMETRIC CONSTRUCTIONS OF COCYCLES

In this section we explain some concrete examples of 2|1-dimensional effective field theories: a Bott element in  $\text{KMF}^{-24}(\text{pt})$ , classes coming from the string orientation for a family of string manifolds over  $X$ , and classes related to the moonshine module and monster group.

**5.1. 24-periodicity and a Bott element.** The 2-periodicity of K-theory together with the 24-periodicity of  $\text{TMF} \otimes \mathbb{C}$  combine to give a 24-periodicity for KMF. In this subsection, we give a geometric representative for the Bott element

$$(17) \quad \beta \in 2|1\text{-EFT}_{\text{eff}}^{-24}(\text{pt}) \rightarrow \widehat{\text{KMF}}^{-24}(\text{pt}),$$

that implements this periodicity.

The line bundle  $\mathcal{L}$  defined in the previous section has period 24 in the sense that its 24<sup>th</sup> tensor power is trivializable. This can be deduced abstractly from the fact that the abelianization of  $\text{MP}_2(\mathbb{Z})$  is  $\mathbb{Z}/24$ . The discriminant modular form  $\Delta \in \text{MF}^{-24}$  realizes this trivialization: it defines a nonvanishing section of  $\mathcal{L}^{\otimes -24}$ .

Our candidate Bott element is the representation of  $\text{Ann}_0^{2|1}(\text{pt})$  with  $E(n) = 0$  for  $n \neq 1$ ,  $E(1) = \mathbb{C}$ , and  $\mathbb{A}(1) = 0$ . The character of this representation is  $q$ . Since there are no nonzero odd differential forms on  $\text{pt}$ , this uniquely determines a cocycle  $\beta \in 2|1\text{-AFT}_{\text{eff}}(\text{pt})$ . The normalized partition function of  $\beta$  is the modular discriminant,  $\Delta \in \text{TMF}^{-24}(\text{pt}) \otimes \mathbb{C}$ , since

$$q \cdot \Phi(q)^{24} = \left( q^{1/24} \prod_n (1 - q^n) \right)^{24} = \eta(q)^{24} = \Delta,$$

where  $\eta$  is the Dedekind  $\eta$ -function. Pulling back  $\beta$  along  $X \rightarrow \text{pt}$  defines an element  $\beta \in 2|1\text{-EFT}_{\text{eff}}^{-24}(X)$  for all  $X$ . Multiplication by  $\beta$  is invertible because the analogous representation,  $\beta^{-1}$ , with  $E(-1) = \mathbb{C}$  can be viewed as having degree +24 (since  $\Delta^{-1} \in \text{MF}^{24}$ ), and the tensor product of these gives the unit representation as a degree zero effective field theory.

If we consider cocycles in  $2|1\text{-EFT}_{\text{eff}}^{+}(X)$ , i.e., those whose underlying representation of  $\text{Ann}_0^{2|1}(X)$  is of *positive* energy, then the class  $\beta$  still exists but is not invertible: the

character of the inverse representation has is a negative power of  $q$ . Indeed,  $\text{Kmf}$  is not 24 periodic, so this is as it should be.

*Remark 5.1.* If we don't build in the Morita theory of subsection 4.1 and instead consider categories of  $\text{Fer}(2n)$ -twisted field theories, then the cocycle  $\beta$  is also encodes the natural isomorphism  $\text{Fer}(24) \simeq \text{Fer}(0)$  that on objects is the bimodule implementing the Morita equivalence between the underlying algebras. Since  $\text{TMF}$  is  $24^2$  periodic, there has been considerable effort in understanding the whether an equivalence  $\text{Fer}(24n) \simeq \text{Fer}(0)$  exists for some  $n > 1$  for various definitions of 2-dimensional field theories, e.g., see [ST11] Section 6 and [DH11].

**5.2. The Witten genus as an effective field theory.** For a vector bundle  $V \rightarrow M$ , let  $S_{q^k}V$  denote the total symmetric power

$$S_{q^k}V := \mathbb{C} \oplus q^k V \oplus q^k S^2 V \oplus \cdots \oplus q^{kl} S^l V \oplus \cdots$$

as a formal sum of vector bundles on  $M$ . For  $M$  even dimensional, compact and spin, the Witten genus of  $M$  is

$$\text{Wit}(M) := \text{Ind}(\not{D} \otimes \bigotimes_{k=1}^{\infty} S_{q^k} T M_{\mathbb{C}}) = \sum_{q \geq 0} q^k \text{Ind}(\not{D} \otimes R_k) \in \mathbb{Z}[[q]] \cong \text{K}(\text{pt})[[q]].$$

where  $R_k$  the vector bundle that is the coefficient of  $q^k$  in the formal sum

$$\bigotimes_{k=1}^{\infty} S_{q^k} T M_{\mathbb{C}} \cong \mathbb{C} \oplus q T M \oplus q^2 (T M \oplus S^2(T M)) \oplus \cdots$$

This description of the Witten genus can be repackaged as an effective field theory in  $2|1\text{-AFT}_{\text{eff}}(\text{pt})$ : for each twisted Dirac operator  $\not{D} \otimes R_k$ , choose a cutoff eigenvalue and thereby extract a finite-dimensional vector space  $E(n)$  with an odd operator  $\mathbb{A}(n) \in \text{End}(E(n))^{\text{odd}}$ . Since  $M$  is compact, the spectrum of the square of the Dirac operator is discrete, and so such cutoffs exist. Over the point, the Chern–Simons form that measures the affect on the Chern character of this cutoff is zero. Hence, choosing cutoffs for all  $n$  we obtain a positive energy representation of  $\text{Ann}_0^{2|1}(\text{pt})$ , and together with the zero differential form we define an object  $\sigma_M \in 2|1\text{-AFT}_{\text{eff}}(\text{pt})$ . By the Atiyah–Singer index theorem, the partition function is

$$Z(\sigma_M) = \left\langle \hat{A}(M) \cdot \text{Ch} \left( \bigotimes_{k=1}^{\infty} S_{q^k} T M_{\mathbb{C}} \right), [M] \right\rangle = \text{ch}(\text{Wit}(M)) \in \mathbb{Q}[[q]] \subset \Omega_{\text{cl}}^{\text{ev}}(\text{pt})[[q]].$$

This defines a Hirzebruch genus associated to the characteristic series

$$\frac{z/2}{\sinh(z/2)} \prod_{n \geq 1} \frac{1}{(1 - q^n e^{z/2})(1 - q^n e^{-z/2})} = \frac{1}{\Phi(q)^2} \exp \left( \sum_{k \geq 1} \frac{E_{2k}(q)}{2k} z^{2k} \right)$$

where  $E_{2k}$  denotes the  $2k^{\text{th}}$  Eisenstein series. Zagier [Zag86] proved the equality above, and it makes evident the modularity properties of the Witten genus:  $E_{2k}$  is modular of weight  $2k$  for  $k \geq 2$ , but  $E_2$  is *not* modular. This puts a condition on the Pontryagin classes of  $M$  if  $Z(\sigma_M) \cdot \Phi^{2d}$  is to be a modular form. Specifically, by the splitting principle we have

$$(18) \quad Z(\sigma_M) = \Phi(q)^{-2d} \left\langle \exp \left( \sum_{k \geq 1} \frac{E_{2k}(q) \text{ch}_k(T M_{\mathbb{C}})}{2k} \right), [M] \right\rangle$$

where  $\text{ch}_k$  denotes the  $4k^{\text{th}}$  component of the Chern character. Equating the 4th degree piece with the first Pontryagin class, we find the if  $M$  has a rational string structure (i.e.,  $p_1(M) = 0$ ), then

$$\Phi^{2d} \cdot Z(\sigma_M) \in \text{MF}^{-2d} = \text{MF}_{2d},$$

and  $\sigma_M$  defines a cocycle in  $2|1\text{-EFT}_{\text{eff}}^{-2d}(\text{pt})$ .

*Remark 5.2.* If we take the *kernels* of the twisted Dirac operators  $\not{D} \otimes R_k$  for all  $k$ , then the resulting effective field theory is *conformal*: it is invariant under the RG-flow (compare Remark 3.6). Hence, the 2-dimensional analog of the kernel of the Dirac operator is a conformal field theory in this (nonstandard) sense.

**5.3. From families of string manifolds to effective field theories over  $X$ .** Given a proper geometric family of spin manifolds  $M \rightarrow X$  (e.g., see [Fre87] Section 1), let  $T_{\mathbb{C}}(M/X) \rightarrow M$  denote the fiberwise complexified tangent bundle. The vector bundles  $S_{q^k}(T_{\mathbb{C}}(M/X))$  on  $M$  can be used to twist the fiberwise spinor bundle, defining a sequence of twisted Dirac operators over  $X$  that determines a class

$$(19) \quad \text{Ind}(\not{D} \otimes \bigotimes_{k=1}^{\infty} S_{q^k} T_{\mathbb{C}}(M/X)) = \sum_{q \geq 0} q^k \text{Ind}(\not{D} \otimes R_k) \in K(X)[[q]].$$

where, as before, we define  $R_k$  as the vector bundle with coefficient  $q^k$ . We refine this class to an effective annular field theory using a result of Mischenko–Fomenko [MF79] as formulated by Freed and Lott [FL10]. We combine their Lemma 7.11 and Equations 7.23 and 7.24 into the following.

**Lemma 5.3** (Freed–Lott). *Given a proper family of geometric spin manifolds  $M \rightarrow X$  and a vector bundle  $V \rightarrow M$  with connection, there is a finite-dimensional smooth subbundle  $E$  of the  $V$ -twisted fiberwise spinor bundle that contains the fiberwise kernel of the twisted Dirac operator. Moreover,  $E$  has a super connection  $\mathbb{A}$  whose degree zero piece is the restriction of the fiberwise Dirac operator to  $E$  and there is a class  $\eta \in \Omega^{\text{odd}}(X)/d\Omega^{\text{ev}}(X)$  with*

$$d\eta = \text{Ch}(\text{Ker}(\not{D})) - \text{Ch}(\mathbb{A})$$

where  $\text{Ch}(\text{Ker}(\not{D}))$  is the differential form-valued Chern character gotten from the Bismut super connection for the family of twisted Dirac operators.

A representative of  $\eta$  comes from the Cheeger–Simons eta form, which can be thought of an infinite-dimensional version of a Chern–Simons form associated to the infinite limit of the RG-flow. Interpreting the original family of spin manifolds as giving rise to a family of (non-effective) field theories, the eta form results from integrating out the higher energy states in the complement of  $E$ .

We apply the lemma above to  $V = R_k$  for  $k \in \mathbb{N}$ . This yields a sequence of vector bundles  $E(k)$  on  $X$  with superconnections  $\mathbb{A}(k)$ , which defines a positive energy representation of  $\text{Ann}_0^{2|1}(X)$ . The associated eta forms  $\eta(k)$  define a class in  $\Omega^{\text{odd}}/d\Omega^{\text{ev}}[[q]]$ , which allows us to promote this representation to an effective annular field theory  $(\sigma_M, \eta) \in 2|1\text{-AFT}_{\text{eff}}(X)$ .

Returning to the notation of Lemma 5.3, the family index theorem gives

$$\text{Ch}(\text{Ind}(\not{D})) = \int_{M/X} \hat{A}(M/X) \text{Ch}(V) = \text{Ch}(\mathbb{A}) + d\eta$$

where the Chern character of the family of Dirac operators is defined using the Bismut super connection of the given geometric data, and  $\hat{A}(M/X)$  is the  $\hat{A}$ -polynomial of the complexified vertical tangent bundle. Letting  $V$  range over  $R_k$  in equation 19, we obtain an expression for the effective partition function

$$Z(\sigma_M, \eta) = \sum_{n \geq 0} q^n (\text{Ch}(\mathbb{A}(n)) + d\eta(n)) = \sum_{n \geq 0} q^n \int_{M/X} \hat{A}(M/X) \text{Ch}(R_k) \in \Omega^{\text{ev}}(X)[[q]].$$

A different expression of this power series in  $q$  arises from the families-version of formula 18

$$Z(\sigma_M, \eta) = \Phi(q)^{-2d} \int_{M/X} \exp \left( \sum_{k \geq 1} \frac{E_{2k}(q) \text{ch}_k(T_{\mathbb{C}}(M/X))}{2k} \right)$$

where  $2d$  is the dimension of the fiber for the map  $M \rightarrow X$ . This expression makes clear the potential lack of modularity from  $p_1(T_{\mathbb{C}}(M/X)) \in \Omega_{\text{cl}}^4(M)$ .

A rational string structure for the family is  $H \in \Omega^3(M)$  with  $dH = p_1(T_{\mathbb{C}}(M/X))$ . We can use such a 3-form to modify the effective partition function (by changing  $\eta$ ) so that the resulting effective field theory is modular of the desired degree: we simply add the odd form

$$\eta_H := -\Phi^{-2d} \int_{M/X} \exp(E_2(q)H) \in (\Omega^{\text{odd}}(X)/d\Omega^{\text{ev}}(X)) \llbracket q \rrbracket.$$

Then

$$\begin{aligned} \Phi^{2d} \cdot Z(\sigma_M, \eta + \eta_H) &= \Phi^{2d} \sum q^n (\text{Ch}(\mathbb{A}(n)) + d\eta(n)) - d\eta_H \\ &= \int_{M/X} \exp \left( \sum_{k \geq 1} \frac{E_{2k}(q) \text{ch}_k(T_{\mathbb{C}}(M/X))}{2k} \right) - d \int_{M/X} \exp(E_2(q)H) \\ &= \int_{M/X} \exp \left( \sum_{k \geq 1} \frac{E_{2k}(q) \text{ch}_k(T_{\mathbb{C}}(M/X))}{2k} \right) - \int_{M/X} \exp(E_2(q)dH) \\ &= \int_{M/X} \exp \left( \sum_{k \geq 2} \frac{E_{2k}(q) \text{ch}_k(T_{\mathbb{C}}(M/X))}{2k} \right) \in \bigoplus_{i+j=-d} \Omega_{\text{cl}}^{2i}(X; \text{MF}^{2j}) \end{aligned}$$

where we have used compatibility between fiber integration and the de Rham operator in the third equality and  $p_1(T_{\mathbb{C}}(M/X)) = dH$  in the fourth. Since the effective partition function is modular, we have produced a cocycle in  $2|1\text{-EFT}_{\text{eff}}^{-2d}(X)$ . Notice that changing the rational string structure  $H$  by an exact form gives the same effective field theory, but modifications by closed forms can give a non-isomorphic theories.

**5.4. Classes associated to the moonshine module.** Let  $\mathbb{M}$  denote the Fischer–Griess monster group. The *moonshine module* is a sequence of representations  $V(k)$  of  $\mathbb{M}$  such that

$$\sum_{k \in \mathbb{Z}} \dim(V(k)) = j(\tau) - 744 = \frac{1}{q} + 196884q + 21493760q^2 + \dots$$

where  $j(\tau)$  is Klein’s  $j$ -invariant, a modular form of weight zero. From this description, the monster module defines a cocycle in  $2|1\text{-EFT}_{\text{eff}}^0(\text{pt})$ , where we take the sequence of vector spaces  $V(k)$  with the *zero* operators  $\mathbb{A}(k) = 0$ .

*Remark 5.4.* The degree zero coefficients,  $\text{KMF}^0(\text{pt}) \cong \mathbb{Z}[x]$ , form a polynomial algebra on a single generator  $x$  that we can take to be the image of the moonshine module. Furthermore, the map  $\text{TMF}^0(\text{pt}) \rightarrow \text{KMF}^0(\text{pt})$  is an isomorphism. Since  $\text{KMF}^0(\text{pt})$  acts on  $\text{KMF}^\bullet(X)$  this suggests that the (differential) geometry of the moonshine module plays a prominent role in the (differential) geometry of  $\text{KMF}$ , and ultimately, any candidate geometric model of  $\text{TMF}$ .

We can use isomorphisms of cocycles to perform a clutching construction for the moonshine cocycle: for a cover  $\{U_i\}$  of  $X$ , choose smooth maps on overlaps  $U_i \cap U_j \rightarrow \text{GL}(V(k))$  for each  $k$  satisfying the usual condition  $U_i \cap U_j \cap U_k$ . This gives a bundle on  $X$  whose fiber is the moonshine module. Although this defines a class in  $2|1\text{-AFT}_{\text{eff}}(X)$ , modularity properties are typically spoiled by such a construction since the curvature of the associated sequence of vector bundles need not define a class in  $\text{TMF}(X) \otimes \mathbb{C}$ . Hence, this clutching construction gives a cocycle in  $2|1\text{-AFT}_{\text{eff}}(X)$  that typically does not lift to  $2|1\text{-EFT}_{\text{eff}}^0(X)$ .

We do obtain the degree 0 modularity property if we reduce the structure group of the bundle to the *monster group*. In more detail, let  $P \rightarrow X$  be a principal bundle with structure group  $\mathbb{M}$ . We can form the associated bundles

$$E(k) := P \times_{\mathbb{M}} V(k),$$

and they come equipped with associated connections which are flat because  $\mathbb{M}$  is discrete. This sequence of vector bundles and flat connections again defines an element

of  $2|1\text{-AFT}_{\text{eff}}(X)$ , but now it also defines a class in  $2|1\text{-EFT}_{\text{eff}}^0(X)$ : flatness of the bundle implies the partition function is  $j(q) - 744 \in \Omega_{\text{cl}}^0(X)[[q]][q^{-1}]$ . Hence, we obtain a cocycle

$$(\mathbb{M}(P), 0) \in 2|1\text{-EFT}_{\text{eff}}^0(X).$$

*Remark 5.5.* We find this striking because the monster group comprises the automorphisms of  $\oplus_k V(k)$  as a *2-dimensional conformal field theory*:  $\mathbb{M}$  is precisely the automorphism group of the vertex operator algebra studied by Borchers. Restricting the isomorphisms in our category of cocycles to preserve this additional vertex operator algebra structure leads to clutching data for which modularity properties are *automatic*.

## APPENDIX A. BACKGROUND MISCELLANY

**A.1. Supermanifolds.** A *supermanifold* is a smooth manifold equipped with a sheaf of  $\mathbb{Z}/2$ -graded complex algebras locally isomorphic to the sections of an exterior bundle; these are called *cs-manifolds* in Deligne and Morgan's review [DM99], and apart from this terminological difference we follow their conventions.

A *vector bundle* over a supermanifold  $M$  is a  $\mathbb{Z}/2$ -graded finitely generated projective module over  $C^\infty(M)$ . Often we think of vector bundles in terms of some total space  $E$ , and then we use the notation  $\Gamma(E)$  for the module over  $C^\infty(M)$  defining  $E$ ; note the vector space  $\Gamma(E)$  is typically quite different from the literal sections  $s: M \rightarrow E$ .

An important supermanifold for us is odd tangent bundle,  $\pi TX$  of an ordinary manifold  $X$ . An  $S$ -point of  $\pi TX$  is given by a pair  $(x, \psi)$  where  $x: S \rightarrow X$  is a map of super manifolds and  $\psi$  is a section of  $x^* \pi TX$ ; equivalently,  $x$  determines a morphism of algebras  $C^\infty(X) \rightarrow C^\infty(S)$  and  $\psi$  determines an odd derivation  $C^\infty(X) \rightarrow C^\infty(S)$  with over  $x$ . Functions on  $\pi TX$  are (complex-valued) differential forms,  $C^\infty(\pi TX) \cong \Omega^\bullet(X)$ . We frequently use the isomorphism

$$\pi TX(S) \cong \mathbf{SMfld}(\mathbb{R}^{0|1}, X)(S) := \mathbf{SMfld}(S \times \mathbb{R}^{0|1}, X)$$

where  $\mathbf{SMfld}(N, M)$  is the presheaf of sets on super manifolds defined by  $S \mapsto \mathbf{SMfld}(S \times N, M)$ . The above bijections show  $\mathbf{SMfld}(\mathbb{R}^{0|1}, X)$  is a representable presheaf,  $\pi TX \cong \mathbf{SMfld}(\mathbb{R}^{0|1}, X)$ , e.g., see [HKST11] for a proof. Furthermore, by this description the de Rham operator—as an odd vector field on  $\pi TX$ —is the derivative at the identity of the  $\mathbb{R}^{0|1}$ -action on  $\mathbf{SMfld}(\mathbb{R}^{0|1}, X)$  gotten from precomposition with the action of  $\mathbb{R}^{0|1}$  on itself by translations. Explicitly, this action is

$$\begin{aligned} \Omega^\bullet(X) \cong C^\infty(\pi TX) &\rightarrow C^\infty(\mathbb{R}^{0|1} \times \pi TX) \cong \Omega^\bullet(X)[\theta], \\ \omega &\mapsto \omega + \theta d\omega, \quad \omega \in \Omega^\bullet(X), \end{aligned}$$

where  $\theta$  is an odd function on  $\mathbb{R}^{0|1}$ , and  $d$  is the de Rham operator.

**A.2. Super Euclidean translation groups.** Consider  $\mathbb{R}^{1|1}$  with the super group structure

$$\mathbb{R}^{1|1}(S) \times \mathbb{R}^{1|1}(S) \rightarrow \mathbb{R}^{1|1}(S), \quad (t, \theta) \cdot (s, \eta) = (t + s + \theta\eta, \theta + \eta),$$

for  $(t, \theta), (s, \eta) \in \mathbb{R}^{1|1}(S)$ . There  $S$ -points are akin to coordinates on  $\mathbb{R}^{1|1}$ . More carefully,  $t \in \mathbb{R}(S)^{\text{ev}} \cong C^\infty(S)^{\text{ev}}$  and  $\theta \in \mathbb{R}(S)^{\text{odd}} \cong C^\infty(S)^{\text{odd}}$ , and so together  $(t, \theta)$  determines a map  $S \rightarrow \mathbb{R}^{1|1}$ . When we write  $(t, \theta) \in \mathbb{R}^{1|1}(S)$ , we will also impose a reality condition on  $t$ : upon taking the quotient by the ideal generated by nilpotent elements in  $S$ , we require the image of  $t$  is a conjugation-invariant function on the underlying reduced manifold of  $S$ . Another perspective on the above identifies  $C^\infty(\mathbb{R}^{1|1}) \cong C^\infty(\mathbb{R})[\theta]$ , and then for a choice of (real) coordinate  $t$  on  $\mathbb{R}$  a map  $S \rightarrow \mathbb{R}^{1|1}$  determines a pair of functions on  $S$  from the pullback of the functions  $t$  and  $\theta$  on  $\mathbb{R}^{1|1}$ .

Give  $\mathbb{R}^{2|1}$  the super group structure

$$\mathbb{R}^{2|1}(S) \times \mathbb{R}^{2|1}(S) \rightarrow \mathbb{R}^{2|1}(S), \quad (z, \bar{z}, \theta) \cdot (w, \bar{w}, \eta) = (z + w, \bar{z} + \bar{w} + \theta\eta, \theta + \eta)$$

for  $(z, \bar{z}, \theta), (w, \bar{w}, \eta) \in \mathbb{R}^{2|1}(S)$ . We have similar descriptions of these coordinates in this case; for example, we can identify  $C^\infty(\mathbb{R}^{2|1}) \cong C^\infty(\mathbb{R}^2)[\theta] \cong C^\infty(\mathbb{C})[\theta]$ , and then choose

coordinates  $(z, \bar{z})$  on  $\mathbb{C}$ . We caution that when these are pulled back along an  $S$ -point  $S \rightarrow \mathbb{C}$ , the associated functions on  $S$  aren't conjugate; this would require a real structure on  $S$ . Instead, after taking the quotient of the structure sheaf of  $S$  by the ideal of nilpotents, the pullbacks of the functions  $z$  and  $\bar{z}$  define conjugate functions on the ordinary underlying (real) manifold of  $S$ . Our reality condition on  $(t, \theta) \in \mathbb{R}^{1|1}(S)$  leads to a map  $\mathbb{R}^{1|1}(S) \rightarrow \mathbb{R}^{2|1}(S)$ ,  $(t, \theta) \mapsto (t, t, \theta)$  that is natural in  $S$  and lifts the inclusion of the real axis  $\mathbb{R} \subset \mathbb{C}$  to these supermanifolds.

There is an action by  $U(1) \cong \text{Spin}(2)$  on  $\mathbb{R}^{2|1}$  that is compatible with the group structure, namely,

$$e^{i\alpha} \cdot (z, \bar{z}, \theta) = (e^{i2\alpha}z, e^{-i2\alpha}\bar{z}, e^{-i\alpha}\theta), \quad e^{i\alpha} \in U(1)(S), \quad (z, \bar{z}, \theta) \in \mathbb{R}^{2|1}(S).$$

The semi-direct product  $\mathbb{R}^{2|1} \rtimes \text{Spin}(2)$  is the 2|1-dimensional *super Euclidean isometry group*.

*Remark A.1.* Stolz and Teichner define a model geometry from the action of  $\mathbb{R}^{2|1} \rtimes \text{Spin}(2)$  (the *isometries*) on  $\mathbb{R}^{2|1}$  (the *model space*). This yields fibered categories of super Euclidean manifolds, and our super annuli are a fibered subcategory.

**A.3. Lie categories.** A *super Lie category* is a category internal to super manifolds, and a *super Lie groupoid* is a groupoid internal to super manifolds. We will often omit the adjective “super.” For a Lie category  $\mathcal{C}$ , we denote the objects by  $\text{Ob}(\mathcal{C})$ , the morphisms by  $\text{Mor}(\mathcal{C})$ , the source and target maps by  $s, t: \text{Mor}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{C})$  respectively, and the unit map by  $u: \text{Ob}(\mathcal{C}) \rightarrow \text{Mor}(\mathcal{C})$ . We further require certain pullbacks to exist for composition to make sense internal to supermanifolds.

A smooth functor  $F: \mathcal{C} \rightarrow \mathbf{Vect}$  consists of a super vector bundle  $V \rightarrow \text{Ob}(\mathcal{C})$  and a morphism of super vector bundles  $s^*V \rightarrow t^*V$  over  $\text{Mor}(\mathcal{C})$  compatible with composition and units. Sometimes we reformulate the data of  $F: \mathcal{C} \rightarrow \mathbf{Vect}$  using the functor of points: for a test super manifold  $S$ ,  $F$  assigns to every  $S$ -point of  $\text{Ob}(\mathcal{C})$  a vector bundle on  $S$ , and to every  $S$ -point of  $\text{Mor}(\mathcal{C})$  a morphism of vector bundles. This description is equivalent by the Yoneda lemma: there are universal  $S$ -families for  $S = \text{Ob}(\mathcal{C})$  and  $S = \text{Mor}(\mathcal{C})$ .

**Example A.2.** Fix a Lie group  $G$ . A functor  $\text{pt}/G \rightarrow \mathbf{Vect}$  is smooth a  $G$ -representation. More generally, if  $G$  acts on a manifold  $X$ , a functor from the action groupoid  $X//G$  to  $\mathbf{Vect}$  is a  $G$ -equivariant vector bundle on  $X$ .

The category of Lie groupoids, bibundles and bibundle maps gives a presentation of the bicategory of smooth stacks. We occasionally use this stacky language, but all maps of stacks are presented in terms of homomorphisms of Lie groupoids.

**A.4. Super connections and Chern–Simons forms.** Let  $E \rightarrow X$  be a super vector bundle. A (*Quillen*) *super connection* [Qui85] on  $E$  is an odd linear map

$$\mathbb{A}: \Omega^\bullet(X; E) \rightarrow \Omega^\bullet(X; E)$$

that satisfies the Leibniz rule

$$\mathbb{A}(\omega \cdot s) = d\omega \cdot + (-1)^{|\omega|} \mathbb{A}(s), \quad s \in \Omega^\bullet(X; E), \quad \omega \in \Omega^\bullet(X).$$

For super vector bundles with super connection  $(E, \mathbb{A}^E)$ ,  $(E, \mathbb{A}^{E'})$  on  $X$  and an isomorphism  $\phi: E \rightarrow E'$ , a path  $\mathbb{A}(\lambda)$  in the space of super connections with  $\mathbb{A}(0) = \mathbb{A}^E$  and  $\mathbb{A}(1) = \mathbb{A}^{E'}$  defines a *Chern–Simons form*

$$\text{CS}(\mathbb{A}^E, \mathbb{A}^{E'}) = \int_{X \times I/X} \text{Ch}(\mathbb{A}(\lambda)) = \int_{X \times I/X} \text{Tr}(e^{-\mathbb{A}(\lambda)^2})$$

where the integral is over the fibers of the projection  $X \times I \rightarrow X$ . A different choice of path changes the integral by an exact form, so the the data of the isomorphism  $\phi$  gives a well-defined class  $\text{CS}(\mathbb{A}^E, \mathbb{A}^{E'}) \in \Omega^{\text{odd}}(X)/d\Omega^{\text{ev}}(X)$ . This class measures the difference of the differential form valued Chern character:

$$\text{Ch}(\mathbb{A}^E) - \text{Ch}(\mathbb{A}^{E'}) = d\text{CS}(\mathbb{A}^E, \mathbb{A}^{E'}).$$

**A.5. Modular forms.** The *degree  $2k$  modular forms*, denoted  $\mathrm{MF}_{2k}$ , are the set of holomorphic functions on  $\mathfrak{h} \subset \mathbb{C}$  satisfying

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau), \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \quad \tau \in \mathfrak{h},$$

and we define the degree  $(2k+1)$  modular forms to consist of the zero function,  $\mathrm{MF}_{2k+1} = \{0\}$ . Multiplication of functions assembles these abelian groups into a graded ring. We define  $\mathrm{MF}^\bullet = \mathrm{MF}_{-\bullet}$  to be the ring with the reversed grading. The action by the subgroup

$$\mathbb{Z} \hookrightarrow \mathrm{SL}_2(\mathbb{Z}), \quad \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

allows us to identify a modular form with a function on the cylinder  $\mathfrak{h}/\mathbb{Z}$  with properties. This permits a Fourier expansion called the  *$q$ -expansion*,  $\mathrm{MF} \rightarrow \mathbb{C}[[q]][q^{-1}]$ , where  $q = e^{2\pi i\tau}$ . Let  $\mathrm{mf} \subset \mathrm{MF}$  denote the subring of those modular forms whose  $q$ -expansion has nonnegative powers of  $q$ , or (equivalently) the subring of  $\mathrm{MF}$  whose elements are holomorphic at  $\tau \rightarrow i\infty$ . Often  $\mathrm{mf}$  is called the ring of modular forms and  $\mathrm{MF}$  is called the ring of *weak* modular forms. We have

$$\pi_* \mathrm{Tmf} \otimes \mathbb{C} \cong \mathrm{mf}_*, \quad \pi_* \mathrm{TMF} \otimes \mathbb{C} \cong \mathrm{MF}_*,$$

and so

$$\mathrm{Tmf}^k(M) \otimes \mathbb{C} \cong \bigoplus_{i+j=k} \mathrm{H}^i(M; \mathrm{mf}^j), \quad \mathrm{TMF}^k(M) \otimes \mathbb{C} \cong \bigoplus_{i+j=k} \mathrm{H}^i(M; \mathrm{MF}^j).$$

The *weight* of a degree  $2k$  modular form is  $k$ .

**A.6. Differential (Tate) K-theory.** A differential refinement  $\widehat{h}$  of a cohomology theory  $h$  gives a commutative square

$$\begin{array}{ccc} \widehat{h}(X) & \xrightarrow{u} & h(X) \\ R \downarrow & & \downarrow \mathrm{ch} \\ \Omega_{\mathrm{cl}}(X; h_{\mathbb{C}}) & \longrightarrow & \mathrm{H}(X; h_{\mathbb{C}}). \end{array}$$

where  $h_{\mathbb{C}} = h(\mathrm{pt}) \otimes \mathbb{C}$  is the complexification of the coefficient ring of  $h$ , and  $\Omega_{\mathrm{cl}}(X; h_{\mathbb{C}})$  consists of closed differential forms with values in this ring having total degree zero. The map  $R$  is called the *curvature map* and the map  $u$  takes the underlying cohomology class of a differential cocycle. These maps are compatible when one takes the underlying cohomology class determined by the closed form and the applies the Chern character to the class in  $h^\bullet(X)$ . Hopkins and Singer [HS05] give a universal construction of differential refinements as an appropriate homotopy pullback of the above; in brief, a differential cocycle is a map into a space representing  $h$  and a differential form with coefficients in  $h_{\mathbb{C}}$ , together data showing the images of these are cohomologous in a cochain model for  $\mathrm{H}(X; h_{\mathbb{C}})$ .

In this paper we work with a fixed model for differential K-theory from [Klo08], Section 4.1. For a smooth manifold  $X$ , define a groupoid  $\mathcal{V}(X)$  whose objects are triples,  $(E, \mathbb{A}, \alpha)$  for  $E$  a super vector bundle on  $X$ ,  $\mathbb{A}$  a super connection, and  $\alpha \in \Omega^{\mathrm{odd}}(X)/d\Omega^{\mathrm{ev}}(X)$ . Define a morphism from  $(E, \mathbb{A}, \alpha)$  to  $(E', \mathbb{A}', \alpha')$  to be an isomorphism  $\phi: E \rightarrow E'$  of super vector bundles such that

$$\alpha = \alpha' + \mathrm{CS}(\mathbb{A}, \phi^* \mathbb{A}').$$

This groupoid has a symmetric monoidal structure that on objects is

$$(E, \mathbb{A}, \alpha) \oplus (E', \mathbb{A}', \alpha') = (E \oplus E', \mathbb{A} \oplus \mathbb{A}', \alpha + \alpha'),$$

and is defined in the obvious way on morphisms. Hence, isomorphism classes of objects form a commutative monoid we denote by  $(\pi_0(\mathcal{V}(X)), +)$ . Define an equivalence relation on this monoid generated by  $[V \oplus \pi V, \nabla \oplus \nabla, 0] \sim 0$  for  $(V, \nabla)$  an ordinary (purely even) vector bundle with connection. Then we have a natural isomorphism of abelian groups,

$\widehat{K}(X) \cong \pi_0(\mathcal{V}(X))/\sim$ . In fact, we obtain an isomorphism of rings where multiplication of cocycles is

$$[E, \mathbb{A}, \alpha] \cdot [E', \mathbb{A}', \alpha'] = [E \otimes E', \mathbb{A} \otimes \mathbb{A}', \alpha \wedge \text{Ch}(E') + \text{Ch}(E) \wedge \alpha' + \alpha \wedge d\alpha'].$$

We prove Theorem 1.3 with respect to this model for  $\widehat{K}(X)$ . We obtain a model for differential Tate K-theory via

$$\widehat{K}_{\text{Tate}}(X) := \widehat{K}(X)[[q]][q^{-1}],$$

with the ring structure inherited from  $\widehat{K}(X)$  and  $\mathbb{Z}[[q]][q^{-1}]$ . We define the differential cohomology theory  $\widehat{\text{KMF}}^{2n}$  as the subset of cocycles in  $\widehat{K}(X)[[q]][q^{-1}]$  whose curvature map factors through

$$(20) \quad \Phi: \bigoplus_{i+j=n} \Omega_{\text{cl}}^{2i}(X; \text{MF}^{2j}) \rightarrow \Omega_{\text{cl}}(X)[[q]][q^{-1}]$$

where MF is the graded ring of modular forms, and  $\Phi$  is induced by a map on coefficients that multiplies a modular form by the  $2n^{\text{th}}$  power of the Euler  $\Phi$ -function and then takes the  $q$ -expansion. This modification to the usual  $q$ -expansion comes from geometric considerations we explain in Section 4. Said differently, we have a pullback square

$$\begin{array}{ccc} \widehat{\text{KMF}}^{2n}(X) & \longrightarrow & \widehat{K}_{\text{Tate}}(X) \\ R \downarrow & & \downarrow R \\ \Omega_{\text{cl}}(X; \text{MF}) & \xrightarrow{\Phi} & \Omega_{\text{cl}}(X)[[q]][q^{-1}] \end{array}$$

We remark that there are no interesting homotopies between closed forms (when viewed as a set), so one can view this pullback as a homotopy pullback in the category of stacks on the site of manifolds. We define  $\widehat{\text{Kmf}}$  similarly, using  $\widehat{K}[[q]]$  and the ring mf.

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